Ph125a Final EXAM

Due Friday, December 8, at 5:00 pm in the TA's mailboxes

- Closed book, and you may use only one page of prepared notes (both sides of a single 8.5"x11" sheet of paper).
- Use of computer software is not allowed on this exam.
- The time limit for this exam is five hours.
- ***Number in* (*parentheses*) *indicates points per problem*.**

1NS. Suppose we have two quantum systems A and B. Let the Hilbert space H_A of A be two-dimensional and have given orthonormal basis states $\{|0_A\rangle, |1_A\rangle\}$. Let the Hilbert space H_B of B be seven-dimensional and have given orthonormal basis $\{|0_B\rangle, |1_B\rangle, \cdots, |6_B\rangle\}$. Throughout this problem let

$$\mathbf{O}_A = |0_A\rangle\langle 1_A| + |1_A\rangle\langle 0_A|,$$
$$\mathbf{O}_B = i|1_B\rangle\langle 6_B| - i|6_B\rangle\langle 1_B|.$$

a) (5) Find the eigenvalues and eigenstates of $\mathbf{O}_A$ and $\mathbf{O}_B$.

b) (5) Find the eigenvalues and eigenstates of the joint Hamiltonian operator

$$\mathbf{H}_0 = \varepsilon \mathbf{O}_A \otimes \mathbf{O}_B,$$

where ε is a real parameter with units of energy.

For the next two parts, suppose we apply a perturbation so that the overall Hamiltonian becomes $\mathbf{H} = \mathbf{H}_0 + \mathbf{W}$, with

$$\mathbf{W} = \lambda |0_A\rangle\langle 0_A| \otimes \mathbf{1}^B.$$

Here λ is a real parameter with units of energy and $\lambda \ll \varepsilon$.

c) (10) Find the zeroth-order eigenstates and first-order eigenvalues of $\mathbf{H}$.

2NS. (5) Which of the following are valid density operators for a spin-$\frac{1}{2}$ system?

$$(i)\ \boldsymbol{\rho} = \frac{1}{2}(\mathbf{1} + i\sigma_y),$$
$$(ii)\ \boldsymbol{\rho} = \frac{1}{2}\left(\mathbf{1} + \frac{1}{2}\sigma_x + \frac{1}{2}\sigma_y + \frac{1}{2}\sigma_z\right)$$
$$(iii)\ \boldsymbol{\rho} = \frac{1}{2}(\mathbf{1} + \sigma_y^2),$$
$$(iv)\ \boldsymbol{\rho} = \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z),$$
$$(v)\ \boldsymbol{\rho} = \mathbf{1} + \frac{i}{2}\sigma_x\sigma_y\sigma_z.$$

3NS. Suppose we have two two-level quantum systems A and B. Let H_A be the Hilbert space for system A, with given orthonormal basis states $\{|0_A\rangle, |1_A\rangle\}$. Let H_B be the Hilbert space for system B, with given orthonormal basis states $\{|0_B\rangle, |1_B\rangle\}$. Let the initial state of the AB system be

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{3}}(|0_A\rangle|0_B\rangle + |0_A\rangle|1_B\rangle + |1_A\rangle|1_B\rangle).$$

a) (2) Is $|\Psi_{AB}\rangle$ an entangled state?
b) (3) Assuming the joint state of the AB system is $|\Psi_{AB}\rangle$, compute the reduced density operator for system B alone.
c) (3) Compute the inner product of $|\Psi_{AB}\rangle$ with the singlet state

$$|s\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle|1_B\rangle - |1_A\rangle|0_B\rangle).$$

That is, compute $\langle s|\Psi_{AB}\rangle$.
d) (7) Now suppose the initial state of the AB system corresponds to a mixed ensemble of $|\Psi_{AB}\rangle$ and $|s\rangle$, with equal probabilities. Specify a complete standard measurement such that only two of the outcomes have nonzero probability.

4NS. Consider three pure states for a spin-$\frac{1}{2}$ system specified by the following Bloch vectors:

$$|\Psi_1\rangle \leftrightarrow \vec{v}_1 = (\sin\theta, \cos\theta, 0),$$
$$|\Psi_2\rangle \leftrightarrow \vec{v}_2 = (-\sin\theta, \cos\theta, 0),$$
$$|\Psi_3\rangle \leftrightarrow \vec{v}_3 = (0, 1, 0).$$

Here the vector notation specifies Cartesian components $\vec{v} = (v_x, v_y, v_z)$.
a) (3) Find the (generalized) Bloch vector that corresponds to the density operator for a mixed ensemble of these three states. Assume that the pure states are mixed with equal probabilities.
b) (7) Assume that the initial state of the spin-$\frac{1}{2}$ system corresponds to

$$\boldsymbol{\rho}(0) = \frac{1}{3}(|\Psi_1\rangle\langle\Psi_1| + |\Psi_2\rangle\langle\Psi_2| + |\Psi_3\rangle\langle\Psi_3|),$$

and that the (time-dependent) Hamiltonian is given by

$$\mathbf{H}_{tot}(t) = -\gamma\mathbf{S} \bullet \vec{B}(t),$$

where

$$\vec{B}(t) = B_0\bar{z} + b_1(\cos(\omega_L t)\bar{x} + \sin(\omega_L t)\bar{y}).$$

Here $\bar{x}$, $\bar{y}$, and $\bar{z}$ are unit vectors along the coordinate axes and $\omega_L = -\gamma B_0$ is the Larmor frequency. Compute $\boldsymbol{\rho}(t)$ in the rotating frame, assuming that there is no dissipation. Note that this can be done using “geometrical” reasoning only!
c) (5) Transform $\boldsymbol{\rho}(t)$ back to the stationary frame.

5NS. (10) Suppose we have a quantum system A that lives in a three-dimensional Hilbert space H_A with given orthonormal basis states $\{|0_A\rangle, |1_A\rangle, |2_A\rangle\}$. Let the Hamiltonian for this system be

$$\mathbf{H} = \hbar\Omega\,\mathbf{O}_A^5,$$

where

$$\mathbf{O}_A = |0_A\rangle\langle 1_A| + |1_A\rangle\langle 0_A| - |2_A\rangle\langle 2_A|$$

and Ω is a real parameter with units of radians per second.
Let $\mathbf{\Pi}_Q$ be the projector onto the state

$$|\Psi_Q\rangle = \frac{1}{2}|0_A\rangle + \frac{1}{2}|1_A\rangle + \frac{1}{\sqrt{2}}|2_A\rangle,$$

and let the initial state of A be given by

$$|\Psi(0)\rangle = c_0|0_A\rangle + c_2|2_A\rangle.$$

Derive an exact expression for

$$\langle\mathbf{\Pi}_Q\rangle = \langle\Psi(t)|\mathbf{\Pi}_Q|\Psi(t)\rangle.$$

6NS. (10) Compute the minimum possible value of the uncertainty product $\Delta\mathbf{x}\Delta\mathbf{H}_0,$ where

$$\mathbf{H}_0 = \frac{\mathbf{p}^2}{2m}$$

and **p** is the momentum operator.