Ph125a Solution set #7

1 Problem 1.

Let first calculate the evolution operator.

$$\mathbf{T}(t,0) = \exp\left(i\frac{\gamma B_0 \boldsymbol{\sigma}_z t}{2\hbar}\right) = \exp\begin{pmatrix} i\frac{\omega t}{2} & 0 \\ 0 & -i\frac{\omega t}{2}\end{pmatrix} = \begin{pmatrix} e^{i\frac{\omega t}{2}} & 0 \\ 0 & e^{-i\frac{\omega t}{2}}\end{pmatrix} \tag{1}$$

where

$$\omega = \frac{\gamma B_0}{\hbar} \tag{2}$$

$\mathbf{T}(0,t)$ is inverse of $\mathbf{T}(t,0)$:

$$\mathbf{T}(0,t) = \mathbf{T}^{-1}(t,0) = \begin{pmatrix} e^{-i\frac{\omega t}{2}} & 0 \\ 0 & e^{i\frac{\omega t}{2}}\end{pmatrix} \tag{3}$$

Then direct calculation gives

$$\begin{aligned}\mathbf{S}_u(t) &= \mathbf{T}^{-1}(0,t)\mathbf{S}_x\mathbf{T}^{-1}(t,0) = \mathbf{T}(t,0)\mathbf{S}_x\mathbf{T}(0,t) = \\ &= \begin{pmatrix} e^{i\frac{\omega t}{2}} & 0 \\ 0 & e^{-i\frac{\omega t}{2}}\end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} e^{-i\frac{\omega t}{2}} & 0 \\ 0 & e^{i\frac{\omega t}{2}}\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0\end{pmatrix} = \\ &= \frac{1}{2}\cos\omega t\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} - \frac{1}{2}\sin\omega t\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix} = \cos\omega t\mathbf{S}_x - \sin\omega t\mathbf{S}_y \end{aligned} \tag{4}$$

From (4) we immediately infer that

$$c_x(t) = \cos\omega t \tag{5}$$

$$c_y(t) = -\sin\omega t \tag{6}$$

Let $|\Psi_0\rangle$ be the initial state of the system (at $t = 0$). Time evolution of the state vector is goverened by time-developement operator as

$$|\Psi(t)\rangle = \mathbf{T}(t,0)|\Psi_0\rangle \tag{7}$$

$$\langle\Psi(t)| = \langle\Psi_0|\mathbf{T}(0,t) \tag{8}$$

Then (using (7) and (8))

$$\begin{aligned}\langle\mathbf{S}_u(t)\rangle &= \langle\Psi(t)|\mathbf{S}_u(t)|\Psi(t)\rangle = \langle\Psi_0|\mathbf{T}(0,t)\mathbf{S}_u(t)\mathbf{T}(t,0)|\Psi_0\rangle = \\ &= \langle\Psi_0|\mathbf{S}_x|\Psi_0\rangle = const, \qquad \text{Q.E.D}\end{aligned} \tag{9}$$

2 Problem 2

The problem's simplest if we work – as in lecture – in a *corotating* basis. Of couse, we could solve the problem in the original nonrotating basis (solving a time-dependent DE); in doing so, though, we'd essentially go to a corotating basis anyways.

So define the rotation operator $R = e^{-i\omega t S_z}$, $|\phi\rangle = R\,|\bar{\phi}\rangle$, and $\bar{H} = RHR^{-1}$; we then have a simpler (corotating) problem for $|\bar{\phi}\rangle$:[1]

$$i\partial_t\,|\bar{\phi}\rangle = (\bar{H} + \omega S_z)\,|\bar{\phi}\rangle = -\left[(\gamma B_0 + \omega)\,S_z + \gamma b_1 S_x\right]|\bar{\phi}\rangle$$

The problem is now time-independent. Defining Δ, $\delta = \gamma b_1$, and $\Omega = \sqrt{\Delta^2 + \delta^2}$, we find eigensolutions with frequencies $\mp\Omega/2$;

$$v_\pm = \frac{1}{\sqrt{2\Omega\,(\Omega \mp \Delta)}}\begin{pmatrix}\delta \\ \pm(\Omega \mp \Delta)\end{pmatrix} \qquad \begin{aligned}\begin{pmatrix}0\\1\end{pmatrix} &= \tfrac{1}{\sqrt{2\Omega}}\left(\sqrt{\Omega - \Delta}v_+ - \sqrt{\Omega + \Delta}v_-\right)\\ |\bar{\phi}(t)\rangle &= \tfrac{1}{\sqrt{2\Omega}}\left(\sqrt{\Omega - \Delta}v_+ e^{-i\Omega t/2} - \sqrt{\Omega + \Delta}v_- e^{i\Omega t/2}\right)\end{aligned}$$

Finally, we must extract $a(t)$. One might expect that one would need to first rotate $v_\pm$ with R and then extract; since, however, we're measuring/preparing states along the same direction as the static magnetic field, the problem simplifies:

$$a(t) = \langle +_z\,|R|\,\bar{\phi}\rangle = e^{-i\frac{\omega t}{2}} < +_z\,|\bar{\phi}\rangle = -ie^{-i\frac{\omega t}{2}}\frac{\delta}{\Omega}\sin\frac{\Omega t}{2}$$

Hence the stated result.

2.1 Comments

- **Lecture results:** Many people used computation-intensive results taken directly from lecture. While it's fine to take (once you understand it) the interaction picture (↔rotating frame) for granted, you can in fact make mistakes – though few in fact did – using the longer formulae relating bloch-sphere angles & eigenenergies to hamiltonian parameters (discussed in lecture 11/9, and in cohen-tan.). So if you use complicated formulae you can't immediately derive in the course of an assignment, tell us you're doing so.

- **Rotating Axes**: The original time-dependent $H(t)$ does have instantaneous eigenstates, with eigenvalues of similar form to $\pm\frac{\Omega}{2}$; the eigenvectors rotate, however, and are not solutions of schrodinger's equation. If, however, one assumes they're intrinsically static and time-evolve according to the *rotating-frame* eigenvalues, then one can turn the crank and get the desired result. People who did this were confused.

2.2 Some other methods

Some other manipulations I saw:

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[1] As you should understand from the first problem, the rotation removes the time dependence of the spin term.

Find Time Evolution Operator Instead of expressing the original state in the eigenbasis and using $|\phi(t)\rangle = \sum e^{-iE_j t} < j\,|\phi(0)\rangle$, one could instead find the *general* time evolution operator matrix and insert the original state (all, of course, in the corotating frame).

Most people who did so used $S\Lambda S^{-1} = T$ methods. One could also abstractly exponentiate $\bar{H} = -\frac{1}{2}\begin{pmatrix} \Delta & -\delta \\ -\delta & -\Delta \end{pmatrix}$, as $\bar{H}^2 = \frac{1}{4}\Omega^2\mathbf{1}$. You immediately find $a(t)$ using

$$e^{-i\bar{H}t} = \cos\frac{\omega t}{2} + i\frac{\bar{H}}{\Omega}\sin\frac{\omega t}{2}$$

Bloch vector DE In class, you proved that the DE for states is equivalent to a DE for the bloch spin vector $< S >$. Using that DE and the correspondance to spin, the problem's purely geometric/trigonometric.

Time-dependent DE A fair number of people simply solved the time-dependent DE. While those people who did so usually got the right relative magnitudes, a fair number missed the overall normalization (remember you must match to b(0)=1) or simply drew their normalization from thin air.

3 Problem 3: Explicit Pauli Method

Let

$$\boldsymbol{\rho}_{ss} = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix} \tag{10}$$

where a is a real number. Here I use the hermiticity and normalization condition of density matrix. For the steady state, the master equation is reduced to

$$i[\boldsymbol{\sigma}_x, \boldsymbol{\rho}_{ss}] + \eta(2\boldsymbol{\sigma}_-\boldsymbol{\rho}_{ss}\boldsymbol{\sigma}_+ - \boldsymbol{\sigma}_+\boldsymbol{\sigma}_-\boldsymbol{\rho}_{ss} - \boldsymbol{\rho}_{ss}\boldsymbol{\sigma}_+\boldsymbol{\sigma}_-) = 0 \tag{11}$$

After routine calculations, we get

$$i\begin{pmatrix} b^* - b & 1-2a \\ 2a-1 & b-b^* \end{pmatrix} + \eta\begin{pmatrix} -2a & -b \\ -c & 2a \end{pmatrix} = 0 \tag{12}$$

That is,

$$i(b - b^*) = -2\eta a \tag{13}$$
$$i(1-2a) = \eta b \tag{14}$$

From the first equation, we can deduce that b is purely imaginary. So let b=ic for some real number c. Then we have

$$c = \eta a \tag{15}$$
$$1 - 2a = \eta c \tag{16}$$

The unique solution to these equations is $a = \frac{1}{\eta^2+2}$,$c = \frac{\eta}{\eta^2+2}$. Therefore

$$\boldsymbol{\rho}_{ss} = \frac{1}{\eta^2+2}\begin{pmatrix} 1 & i\eta \\ -i\eta & \eta^2+1 \end{pmatrix} \tag{17}$$

Now,

$$w = \frac{1}{2}Tr[\boldsymbol{\sigma}_z\boldsymbol{\rho}_{ss}] = -\frac{\eta^2}{2(\eta^2+2)} \tag{18}$$

$$u = \frac{1}{2}Tr[\boldsymbol{\sigma}_x\boldsymbol{\rho}_{ss}] = 0 \tag{19}$$

$$v = \frac{1}{2}Tr[\boldsymbol{\sigma}_y\boldsymbol{\rho}_{ss}] = -\frac{\eta}{\eta^2+2} \tag{20}$$

Comment

We are working consistently in the rotating frame, so we should interpret (w,u,v) according to it.