The Born Series

I wanted to give a brief introduction to the Born series, because I will use the Born approximation to do a quick introductory calculation for scattering in the Coulomb potential. However, that calculation is not extremely important and is only a minor part of this lecture, so if you want you could skip to the next section and return to this part later. In this section, we want to formally construct a solution to the scattering problem using the method of Green's functions. Letting $U = \frac{2m}{\hbar^2}V$ and $k^2 = \frac{2mE}{\hbar^2}$, the Schrödinger equation can be written as:

$$(\nabla^2 + k^2)\Psi = U\Psi$$

Finding a Green's function $G\left(\vec{r},\vec{r}^{\,\prime}\right)$ that is a solution of

$$(\nabla^2 + k^2)G\left(\vec{r},\vec{r}^{\,\prime}\right) = -4\pi\delta\left(\vec{r}-\vec{r}^{\,\prime}\right)$$

produces the integral equation:

$$\Psi_{\vec{k}}\left(\vec{r}\right) = Ne^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int G\left(\vec{r},\vec{r}^{\,\prime}\right)U\left(\vec{r}^{\,\prime}\right)\Psi_{\vec{k}}\left(\vec{r}^{\,\prime}\right)d^3\vec{r}^{\,\prime}$$

The formal solution to this integral equation can be constructed iteratively, giving what is known as the Born series:

$$\begin{aligned}\Psi_{\vec{k}}\left(\vec{r}\right) &= Ne^{i\vec{k}\cdot\vec{r}} - N\frac{1}{4\pi}\int d^3\vec{r}^{\,\prime}G\left(\vec{r},\vec{r}^{\,\prime}\right)U\left(\vec{r}^{\,\prime}\right)e^{i\vec{k}\cdot\vec{r}^{\,\prime}} \\ &\quad + N\left(-\frac{1}{4\pi}\right)^2\int d^3\vec{r}^{\,\prime}\int d^3\vec{r}^{\,\prime\prime}G\left(\vec{r},\vec{r}^{\,\prime}\right)U\left(\vec{r}^{\,\prime}\right)G\left(\vec{r}^{\,\prime},\vec{r}^{\,\prime\prime}\right)U\left(\vec{r}^{\,\prime\prime}\right)e^{i\vec{k}\cdot\vec{r}^{\,\prime\prime}} + \ldots\end{aligned}$$

There are infinitely many solutions to the Green's function equation, but the two we are most interested in are:

$$G_{\pm}\left(\vec{r},\vec{r}^{\,\prime}\right) = \frac{\exp\left[\pm ik\left|\vec{r}-\vec{r}^{\,\prime}\right|\right]}{\left|\vec{r}-\vec{r}^{\,\prime}\right|}$$

known as the retarded (G_+) and advanced (G_-) Green's functions (because they correspond to outgoing and incoming waves repectively). This gives us the integral equations:

$$\begin{aligned}\Psi_{\vec{k}}^{(\pm)}\left(\vec{r}\right) &= Ne^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int G_{\pm}\left(\vec{r},\vec{r}^{\,\prime}\right)U\left(\vec{r}^{\,\prime}\right)\Psi_{\vec{k}}^{(\pm)}\left(\vec{r}^{\,\prime}\right)d^3\vec{r}^{\,\prime} \\ &= Ne^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int\frac{\exp\left[\pm ik\left|\vec{r}-\vec{r}^{\,\prime}\right|\right]}{\left|\vec{r}-\vec{r}^{\,\prime}\right|}U\left(\vec{r}^{\,\prime}\right)\Psi_{\vec{k}}^{(\pm)}\left(\vec{r}^{\,\prime}\right)d^3\vec{r}^{\,\prime}\end{aligned}$$

To get a scattering amplitude from this, we make the following approximation for the $r \to \infty$ limit:

$$\left|\vec{r}-\vec{r}^{\,\prime}\right| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}^{\,\prime} + r^{\prime 2}} \simeq r - \hat{r}\cdot\vec{r}^{\,\prime} + \frac{\left(\hat{r}\times\vec{r}^{\,\prime}\right)^2}{2r} + \ldots$$

and keep up through first order in the exponential term and only zeroth order in the denominator. So the asymptotic limit of the integral equation becomes:

$$\Psi_{\vec{k}}^{(\pm)}\left(\vec{r}\right) \sim Ne^{i\vec{k}\cdot\vec{r}} - \frac{e^{\pm ikr}}{4\pi r}\int\exp\left[\mp ik\hat{r}\cdot\vec{r}^{\,\prime}\right]U\left(\vec{r}^{\,\prime}\right)\Psi_{\vec{k}}^{(\pm)}\left(\vec{r}^{\,\prime}\right)d^3\vec{r}^{\,\prime}$$

and putting it in the asymptotic scattering form gives:

$$\Psi_{\vec{k}}^{(\pm)}(\vec{r}) \sim N\left(e^{i\vec{k}\cdot\vec{r}} + f_{\vec{k}}^{(\pm)}(\vec{r})\,\frac{\exp[\pm ikr]}{r}\right)$$

$$f_{\vec{k}}^{(\pm)}(\hat{r}) = -\frac{1}{4\pi N}\int \exp\left[\mp ik\hat{r}\cdot\vec{r}'\right]U\left(\vec{r}'\right)\Psi_{\vec{k}}^{(\pm)}\left(\vec{r}'\right)d^3\vec{r}'$$
$$= -\frac{m}{2\pi\hbar^2 N}\int \exp\left[\mp ik\hat{r}\cdot\vec{r}'\right]V\left(\vec{r}'\right)\Psi_{\vec{k}}^{(\pm)}\left(\vec{r}'\right)d^3\vec{r}'$$

For the scattering of an incoming plane wave, the first order approximation is known as the Born approximation, and is given by the first iteration (i.e. plugging in $\Psi_{\vec{k}}\left(\vec{r}'\right) = Ne^{i\vec{k}\cdot\vec{r}}$ to the right hand side of the equation):

$$f_{\vec{k}}^{Born}(\hat{r}) = -\frac{m}{2\pi\hbar^2 N}\int \exp\left[-ik\hat{r}\cdot\vec{r}'\right]V\left(\vec{r}'\right)Ne^{i\vec{k}\cdot\vec{r}}d^3\vec{r}'$$
$$= -\frac{m}{2\pi\hbar^2}\int \exp\left[i\left(\vec{k}-k\hat{r}\right)\cdot\vec{r}'\right]V\left(\vec{r}'\right)d^3\vec{r}'$$
$$= -\frac{m}{2\pi\hbar^2}\int \exp\left[i\vec{q}\cdot\vec{r}'\right]V\left(\vec{r}'\right)d^3\vec{r}'$$

where $\vec{q} = \vec{k} - k\hat{r}$, gives $q = 2k\sin\left(\frac{\theta}{2}\right)$ for the scattering angle θ (i.e. the angle between $\hat{k}$ and $\hat{r}$). For a central potential ($V\left(\vec{r}\right) = V(r)$), we can explicitly perform the solid angle integral to obtain:

$$f_k^{Born}(\theta) = -\frac{2m}{\hbar^2}\int_0^\infty V(r')\frac{\sin(qr')}{qr'}r'^2dr'$$

Coulomb Scattering

Scattering in the Coulomb field is an important subject to study simply because electromagnetic interactions are among the most observable physical interactions in nature, particularly at the quantum mechanical level. If physical application was not enough motivation to study it, additional inspiration can be found from the fact that it is on the short list of quantum systems that can be solved exactly. As a preliminary step, consider the cross section for the Coulomb potential $V(r) = \frac{q_1q_2}{r}$ obtained from classical mechanics (i.e. the Rutherford scattering formula):

$$\frac{d\sigma}{d\Omega} = \frac{q_1^2q_2^2}{16E^2\sin^4\left(\frac{\theta}{2}\right)}$$

Clearly, integrating over the entire solid angle gives an infinite total cross section due to the divergence at $\theta = 0$ (i.e. forward scattering). This is indicative of the fact that the $\frac{1}{r}$ potential, though weak enough to vanish as $r \to \infty$, is still strong enough to have a significant effect at all distances. In this sense, it is considered a long range potential and one might suspect that the scattering formalism in quantum mechanics developed for potentials of finite range may not be applicable to case.

The next step is to look at the Coulomb potential in the scattering formalism that we have already seen. Using the Born approximation for a central potential:

$$f_k(\theta) \simeq f_k^{Born}(\theta) = -\frac{2m}{\hbar^2}\int_0^\infty V(r')\frac{\sin(qr')}{qr'}r'^2dr'$$

with $q = 2k\sin\left(\frac{\theta}{2}\right)$, and plugging in the Coulomb potential gives:

$$f_k^{Born}(\theta) = -\frac{mq_1q_2}{\hbar^2 k \sin\left(\frac{\theta}{2}\right)} \int_0^\infty {}' \sin(qr')dr'$$

which is obviously ill-defined, since the integral is not convergent. This is an indication that the Coulomb potential does not nicely fit the standard scattering formalism. However if we wanted, we could salvage our attempt to use the standard formalism by using a regularization trick. Consider the Yukawa potential (or screened Coulomb potential):

$$V(r) = \frac{q_1q_2}{r} e^{-\mu r}$$

Where $\mu > 0$ is a real number, and physically represents the screening effect resulting from the electric charge balance in the universe. (No such charge balance occurs for the gravitational potential which is also a $\frac{1}{r}$ potential, and this is exactly why it is the dominant force on the cosmological scale.) Plugging this into the Born approximation gives:

$$\begin{aligned} f_k^{Born}(\theta) &= -\frac{mq_1q_2}{\hbar^2 k \sin\left(\frac{\theta}{2}\right)} \int_0^\infty {}' \sin(qr') e^{-\mu r} dr' \\ &= -\frac{mq_1q_2}{\hbar^2 k \sin\left(\frac{\theta}{2}\right)} \frac{q}{\mu^2 + q^2} \\ &= -\frac{2mq_1q_2}{\hbar^2 \left(\mu^2 + 4k^2 \sin^2\left(\frac{\theta}{2}\right)\right)} \end{aligned}$$

Notice that this gives a cross-section that is well-defined for all θ and gives a finite total cross-section:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &\simeq \frac{q_1^2 q_2^2}{16E^2} \frac{1}{\left[\sin^2\left(\frac{\theta}{2}\right) + \left(\frac{\mu}{2k}\right)^2\right]^2} \\ \sigma &\simeq \frac{q_1^2 q_2^2}{16E^2} \frac{64\pi k^4}{\mu^4 + 4k^2\mu^2} \end{aligned}$$

(where $E = \frac{\hbar^2 k^2}{2m}$ is the free particle energy) and so this potential should give a convergent Born series. Now the regularization trick is to let $\mu \to 0$, giving:

$$\begin{aligned} f_k^{Born}(\theta) &= -\frac{mq_1q_2}{2\hbar^2 k^2 \sin^2\left(\frac{\theta}{2}\right)} \\ \frac{d\sigma}{d\Omega} &\simeq \frac{q_1^2 q_2^2}{16E^2 \sin^4\left(\frac{\theta}{2}\right)} \end{aligned}$$

This agrees exactly with the Rutherford equation. However, this is a small victory, since there will be a breakdown in the solution in the $\mu \to 0$ limit as the Born series will no longer be convergent.

Even though this regularization is good in the sense that the regulator $e^{-\mu r}$ corresponds to an actual physical effect, I typically am uncomfortable trusting the results obtained from regularization tricks, since it is hard to tell when they actually give sensible results. Here is an interesting example that explains just what I mean: one of the standard regulators (used particularly in quantum field theory) is the Riemann Zeta function $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ which has its value at $s = -1$ defined uniquely by analytic continuation to be $\zeta(-1) = -\frac{1}{12}$. Of course, this implies a result that should make anyone uncomfortable: $\sum_{k=1}^{\infty} k = -\frac{1}{12}$.

Now that we have seen that the classical and the Born approximation cross sections for the Coulomb potential agree, but still lack confidence in the legitamacy of either result, we turn to the Schrödinger equation to try to find a result that is not suspect. We are fortunate since (as mentioned) we can produce an exact solution for scattering in the Coulomb field. This is most naturally treated using parabolic coordinates (since it easily allows one to pick out a solution with the desired scattering asymptotic behavior).

$$\xi = r + z$$
$$\eta = r - z$$
$$\phi = \arctan(y/x)$$

which has the Laplacian given by:

$$\nabla^2 = \frac{4}{\xi+\eta}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{4}{\xi+\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right) + \frac{1}{\xi\eta}\frac{\partial^2}{\partial\phi^2}$$

In these coordinates, the Schrödinger equation becomes:

$$\frac{4}{\xi+\eta}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\Psi\right) + \frac{4}{\xi+\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\Psi\right) + \frac{1}{\xi\eta}\frac{\partial^2}{\partial\phi^2}\Psi - \frac{mq_1q_2}{\hbar^2}\frac{4}{\xi+\eta}\Psi + \frac{2mE}{\hbar^2}\Psi = 0$$

so we can use separation of variables with a wavefunction of the form:

$$\Psi(\xi,\eta,\phi) = f_1(\xi)f_2(\eta)e^{im_l\phi}$$

with $m_l \in \mathbf{Z}$ so as to maintain single-valuedness. Plugging this in gives:

$$\frac{1}{f_1(\xi)}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}f_1(\xi)\right) + \frac{1}{f_2(\eta)}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}f_2(\eta)\right) - \frac{m_l^2}{4}\left(\frac{1}{\xi}+\frac{1}{\eta}\right) - \frac{mq_1q_2}{\hbar^2} + \frac{mE}{2\hbar^2}(\xi+\eta) = 0$$

which separates to:

$$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}f_1(\xi)\right) + \left(-\frac{m_l^2}{4}\frac{1}{\xi} + \frac{mE}{2\hbar^2}\xi + \beta\right)f_1(\xi) = 0$$
$$\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}f_2(\eta)\right) + \left(-\frac{m_l^2}{4}\frac{1}{\eta} + \frac{mE}{2\hbar^2}\eta - \frac{mq_1q_2}{\hbar^2} - \beta\right)f_2(\eta) = 0$$

where β is just a separation constant. If we wanted to, at this point we could solve for the bound state energy eigenfunctions (cf Landau and Lifschitz Ch. 37), and compare these results with the familiar spherical coordinate solution, but I don't think this is particularly enlightening, so I will leave it to those who are interested to do on their own.

Now we want to impose the physical conditions that correspond to the scattering problem. First of all, we know that there should be no ϕ-dependence for scattering off a central potential, so we can let $m_l = 0$. The asymptotic form for a scattering solution is:

$$\Psi_k \sim e^{ikz} \quad \text{for } -\infty < z < 0 \text{ as } r \to \infty$$

which in parabolic coordinates translates to:

$$\Psi_k \sim e^{ik(\xi-\eta)/2} \quad \text{for all } \xi \text{ as } \eta \to \infty$$

giving the desired forms:

$$f_1 \sim e^{ik\xi/2} \quad \text{for all } \xi$$
$$f_2 \sim e^{-ik\eta/2} \quad \text{as } \eta \to \infty$$

Obviously, the first condition necessitates equality:

$$f_1(\xi) = e^{ik\xi/2}$$

which requires $\beta = -i\frac{k}{2}$ in order to satisfy the $f_1(\xi)$ differential equation in the form (with $\frac{\hbar^2 k^2}{2m}$ substituted in for E):

$$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}f_1(\xi)\right) + (k^2\xi + \beta)f_1(\xi) = 0$$

Letting $A = \frac{mq_1q_2}{\hbar^2}$, it remains to solve the equation:

$$\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}f_2(\eta)\right) + \left(k^2\eta - A + i\frac{k}{2}\right)f_2(\eta) = 0$$

To make this easier and satisfy the asymptotic form, let:

$$f_2(\eta) = e^{-ik\eta/2}w(\eta)$$

so that plugging this into the $f_2(\eta)$ differential equation gives:

$$\eta\frac{\partial^2 w(\eta)}{\partial\eta^2} + (1 - ik\eta)\frac{\partial w(\eta)}{\partial\eta} - Aw(\eta) = 0$$

which is the differential equation for a confluent hypergeometric function, giving the solution:

$$w(\eta) = {}_1F_1\left(-i\frac{A}{k}, 1; ik\eta\right)$$

This should not be suprising since the bound state eigenfunctions for the Coulomb potential also involved confluent hypergeometric function. Putting it all together, we have the solution:

$$\Psi_k(\xi,\eta,\phi) = {}_1F_1\left(-i\frac{A}{k}, 1; ik\eta\right)e^{ik(\xi-\eta)/2}$$

Of course, unless you are very familiar with the confluent hypergeometric functions, this is probably fairly meaningless to you in this form. To get something more meaningful out of this, we can use the leading order asymptotic expansion for large pure imaginary z:

$${}_1F_1(\alpha,\gamma;z) \sim \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha} + \frac{\Gamma(\gamma)}{\Gamma(\alpha)}e^z z^{\alpha-\gamma}$$

which gives:

$$\begin{aligned}
\Psi_k(\xi,\eta,\phi) &\sim \left(\frac{\Gamma(1)}{\Gamma\left(1+i\frac{A}{k}\right)}(-ik\eta)^{i\frac{A}{k}} + \frac{\Gamma(1)}{\Gamma\left(-i\frac{A}{k}\right)}e^{ik\eta}(ik\eta)^{-i\frac{A}{k}-1}\right)e^{ik(\xi-\eta)/2} \\
&= \left(\frac{1}{\Gamma\left(1+i\frac{A}{k}\right)}(-ik\eta)^{i\frac{A}{k}} - \frac{1}{\Gamma\left(1-i\frac{A}{k}\right)}\frac{A}{k^2}(ik\eta)^{-i\frac{A}{k}}\frac{e^{ik\eta}}{\eta}\right)e^{ik(\xi-\eta)/2} \\
&= \frac{1}{\Gamma\left(1+i\frac{A}{k}\right)}\exp\left[i\left(k\frac{(\xi-\eta)}{2} + \frac{A}{k}\ln(k\eta) - i\frac{A}{k}\frac{\pi}{2}\right)\right] \\
&\quad - \frac{1}{\Gamma\left(1-i\frac{A}{k}\right)}\frac{A}{k^2}\frac{1}{\eta}\exp\left[i\left(k\frac{(\xi+\eta)}{2} - \frac{A}{k}\ln(k\eta) - i\frac{A}{k}\frac{\pi}{2}\right)\right]
\end{aligned}$$

where I used the gamma function property $\Gamma(n+1) = n\Gamma(n)$. Converting back to cartesian components and choosing a new normalization $N = \Gamma\left(1+i\frac{A}{k}\right)\exp\left[-\frac{A}{k}\frac{\pi}{2}\right]$ (which is of course arbitrary, since these are non-normalizable solutions, but gives a nice asymptotic form), we have:

$$\Psi_k = N_1F_1\left(-i\frac{A}{k}, 1; ik(r-z)\right)e^{ikz}$$

$$\Psi_k \sim \exp\left[ikz + i\frac{A}{k}\ln(k(r-z))\right] - \frac{\Gamma\left(1+i\frac{A}{k}\right)}{\Gamma\left(1-i\frac{A}{k}\right)}\frac{A}{k^2}\frac{\exp\left[ikr - i\frac{A}{k}\ln(k(r-z))\right]}{r-z}$$

This shows why we were unable to apply the familiar scattering formalism in a straightforward manner. Specifically, the logarithmic terms in the phases distort the incident plane wave and the scattered spherical wave even at large distances, preventing a solution with the simple asymptotic scattering form:

$$\Psi_k \sim \exp[ikz] + f_k(\theta)\frac{\exp[ikr]}{r}$$

(If you carried out the asyptotic expansion of the confluent hypergeometric function to another order, you would also find a $\frac{1}{r}$ distortion in the amplitudes, which just adds to the problems encountered when trying to apply the simple scattering formalism.) However, writing the result in the form:

$$\Psi_k \sim \exp\left[ikz + i\frac{A}{k}\ln(kr(1-\cos\theta))\right] + f_k(\theta)\frac{\exp\left[ikr - i\frac{A}{k}\ln(kr)\right]}{r}$$

$$f_k(\theta) = -\frac{\Gamma\left(1+i\frac{A}{k}\right)}{\Gamma\left(1-i\frac{A}{k}\right)}\frac{A}{k^2}\frac{\exp\left[-i\frac{A}{k}\ln(1-\cos\theta)\right]}{1-\cos\theta}$$

and comparing to the simple asymptotic scattering form shows that the difference would only give corrections to the current density that vanish as $r \to \infty$, so we can in fact use this as a bone fide scattering amplitude and derive a cross section and partial wave analysis from it. Using $1-\cos\theta = \sin^2\left(\frac{\theta}{2}\right)$, we obtain the results:

$$f_k(\theta) = -\frac{\Gamma\left(1+i\frac{A}{k}\right)}{\Gamma\left(1-i\frac{A}{k}\right)}\exp\left[-i2\frac{A}{k}\ln\left(\sin\left(\frac{\theta}{2}\right)\right)\right]\frac{A}{k^2}\frac{1}{\sin^2\left(\frac{\theta}{2}\right)}$$

$$= \sum_{l=0}^{\infty}(2l+1)\frac{1}{2ik}\left(\frac{\Gamma\left(l+1+i\frac{A}{k}\right)}{\Gamma\left(l+1-i\frac{A}{k}\right)} - 1\right)P_l(\cos\theta)$$

$$\frac{d\sigma}{d\Omega} = \frac{A^2}{k^4\sin^4\left(\frac{\theta}{2}\right)} = \frac{q_1^2q_2^2}{16E^2\sin^4\left(\frac{\theta}{2}\right)}$$

Remarkably, we have found that the scattering cross section derived from the exact solution agrees exactly with the Rutherford equation obtained classically (and the regularized scattering result as well). In fact in turns out that the Rutherford scattering equation is also obtained (in the non-relativistic limit) from quantum field theory in several different ways. This is actually a pretty amazing situation, finding such stong agreement among all the different physical formulations of a particular interaction.

Of course, just as in the regular scattering formalism, in order to have this scattering solution correspond to an actual physical scattering process, we would have to do something similar to the wave packet construction of incident plane waves. That is to say, in order to have a physical incident state, we would have to take some normalizable superposition of the $\Psi_k = N_1F_1\left(-i\frac{A}{k}, 1; ik(r-z)\right)e^{ikz}$ states, preferably highly localized about some given momentum.

To see how the partial wave analysis is derived, I need to present the spherical solutions of continuum states in the Coulomb potential. (For additional reading on this subject, Messiah's *Quantum Mechanics* provides a good treatment.) For wavefunctions of the form:

$$\psi_k(r,\theta,\phi) = \frac{u_{k,l}(r)}{r} Y_l^m(\theta,\phi)$$

the radial equation

$$\frac{d^2 u_{k,l}(r)}{dr^2} + \left[k^2 - \frac{2A}{r} - \frac{l(l+1)}{r^2} \right] u_{k,l}(r) = 0$$

has the general solution:

$$u_{k,l}(r) = C_1 F_l\left(\frac{A}{k}, kr\right) + C_2 G_l\left(\frac{A}{k}, kr\right)$$

where F_l is regular and G_l is irregular ($\frac{1}{r^l}$ singularity), so we only want to keep the F_l part of this. (If you wanted to look at some special system that does not include the origin then you would want to consider both terms.) F_l, known as the regular Coulomb wavefunction, is defined by:

$$F_l\left(\frac{A}{k}, kr\right) = C_l\left(\frac{A}{k}\right)(kr)^{l+1} \exp[-ikr]\, {}_1F_1\left(l+1-i\frac{A}{k}, 2l+2, 2ikr\right)$$

$$C_l\left(\frac{A}{k}\right) = 2^l \exp\left[-\frac{\pi A}{2k}\right] \frac{\left|\Gamma\left(l+1+i\frac{A}{k}\right)\right|}{(2l+1)!}$$

where $C_l\left(\frac{A}{k}\right)$ is just a conventional normalization. Thus we have the continuum wavefunctions:

$$\psi_k(r,\theta,\phi) = \frac{1}{r} F_l\left(\frac{A}{k}, kr\right) Y_l^m(\theta,\phi)$$

We can also write the general solution of the radial equation in terms of (irregular) solutions that represent incoming and outgoing wave:

$$u_{k,l}(r) = c_1 u_l^{(-)}\left(\frac{A}{k}, kr\right) + c_2 u_l^{(+)}\left(\frac{A}{k}, kr\right)$$

which are given by the following relations:

$$u_l^{(\pm)}\left(\frac{A}{k}, kr\right) = e^{\mp i\delta_l}\left[G_l\left(\frac{A}{k}, kr\right) \pm i F_l\left(\frac{A}{k}, kr\right) \right]$$

The asymptotic forms ($\rho \gg l(l+1) + \gamma^2$) of the spherical Coulomb functions are:

$$F_l(\gamma,\rho) \sim \sin\left(\rho - \gamma \ln 2\rho - \frac{\pi}{2} l + \delta_l\right)$$

$$G_l(\gamma,\rho) \sim \cos\left(\rho - \gamma \ln 2\rho - \frac{\pi}{2} l + \delta_l\right)$$

$$u_l^{(\pm)}(\gamma,\rho) \sim \exp\left[\pm i\left(\rho - \gamma \ln 2\rho - \frac{\pi}{2} l\right)\right]$$

$$\delta_l = \arg\Gamma\left(l+1+i\frac{A}{k}\right) = \frac{\Gamma\left(l+1+i\frac{A}{k}\right)}{\left|\Gamma\left(l+1+i\frac{A}{k}\right)\right|}$$

This is all in complete analogy with the spherical Bessel and Hankel functions of the free Schrödinger equation (expect that the $\frac{1}{r}$ is not included in the definitions of these functions). Expanding the Coulomb scattering solution in terms of Legendre polynomials:

$$\Psi_k = N\, {}_1F_1\left(-i\frac{A}{k}, 1; ik(r-z)\right) e^{ikz}$$

$$= \sum_{l=0}^{\infty} (2l+1) i^l e^{i\delta_l} \frac{F_l\left(\frac{A}{k}, kr\right)}{kr} P_l(\cos\theta)$$

and comparing to the same process we used for partial wave analysis for the plane wave expansion:

$$e^{ikz} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$

$$\exp[ikz] + f_k(\theta)\frac{\exp[ikr]}{r} = \sum_{l=0}^{\infty}(2l+1)i^l e^{i\delta_l}\frac{\sin\left(kr - \frac{\pi}{2}l + \delta_l\right)}{kr}P_l(\cos\theta)$$

$$f_k(\theta) = \sum_{l=0}^{\infty}(2l+1)\frac{(e^{2i\delta_l}-1)}{2ik}P_l(\cos\theta)$$

leads us to the analogous result:

$$\begin{aligned} f_k(\theta) &= \sum_{l=0}^{\infty}(2l+1)\frac{(e^{2i\delta_l}-1)}{2ik}P_l(\cos\theta) \\ &= \sum_{l=0}^{\infty}(2l+1)\frac{1}{2ik}\left(\frac{\Gamma\left(l+1+i\frac{A}{k}\right)}{\Gamma\left(l+1-i\frac{A}{k}\right)} - 1\right)P_l(\cos\theta) \end{aligned}$$

For the case of an attractive potential ($A < 0$), we find scattering resonances at the poles of the corresponding cross section. The singular structure of the gamma function is:

$$\Gamma(z) \sim \frac{1}{z-n} \quad \text{for } z \sim n = 0, -1, -2, \ldots$$

Hence, the resonant energies for Coulomb scattering occur when $i\frac{A}{k} = -1, -2, \ldots$ and so are found (as one should expect) to exactly coincide with the bound state energies of the Coulomb potential:

$$E_{res} = \frac{\hbar^2 k_{res}^2}{2m} = -\frac{m q_1^2 q_2^2}{2\hbar^2 n^2} \quad \text{for } n = 1, 2, \ldots$$