For V a vector space on a field F.
For subsets S_1 and S_2 of V, we write: $S_1 + S_2 = \{|x_1\rangle + |x_2\rangle : |x_1\rangle\epsilon S_1, |x_2\rangle\epsilon S_2\}$.
A subset W of V is a **subspace** of V if W is closed under the operations of the vector space and contains the zero vector $|0\rangle$.

(Notice that for subspaces W_1 and W_2, $W_1 + W_2$ is also a subspace.)

The **orthogonal complement of a subset** W of an inner product vector space space V is:
$W^{\perp} = \left\{|x\rangle\epsilon V : \langle y|x\rangle = 0 \text{ for all } |y\rangle\epsilon W\right\}$
We write: $V = W_1 \oplus W_2$ for the subspaces W_1 and W_2 of V, when $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{|0\rangle\}$.
For an operator $T : V \rightarrow V$, we define:

Range of $T = R(T) = \{T|x\rangle : |x\rangle\epsilon V\}$

Null Space (Kernel) of $T = N(T) = \{|x\rangle\epsilon V : T|x\rangle = |0\rangle\}$

For $V = W_1 \oplus W_2$, the operator P is a **projection on W$_1$ along W$_2$** if for $|x\rangle = |x_1\rangle + |x_2\rangle$ (where: $|x\rangle\epsilon V, |x_1\rangle\epsilon W_1, |x_2\rangle\epsilon W_2$), $P|x\rangle = |x_1\rangle$.
P is a linear operator.
$R(P) = W_1$
$N(P) = W_2$
P is a projection iff $P^2 = P$

A projection P on a inner product vector space V is an **orthogonal projection** if $R(P)^{\perp} = N(P)$ and $N(P)^{\perp} = R(P)$.
P is an orthogonal projection iff P has an adjoint $P^{\dagger}$ and $P^2 = P = P^{\dagger}$.

Proof of P is a projection iff $P^2 = P$:
(i) P projection $\Rightarrow P^2 = P$:

For any $|x\rangle\epsilon V$, we can write $|x\rangle = |x_1\rangle + |x_2\rangle$, with $|x_1\rangle\epsilon W_1, |x_2\rangle\epsilon W_2$. We can also write: $|x_1\rangle = |x_1\rangle + |0\rangle$. Thus,

$P|x\rangle = |x_1\rangle$ and $P^2|x\rangle = P|x_1\rangle = |x_1\rangle$. Since this holds for any $|x\rangle$, $P^2 = P$.

(ii) $P^2 = P \Rightarrow P$ projection:

For any $|x\rangle\epsilon V$, we can write $|x\rangle = P|x\rangle + [|x\rangle - P|x\rangle]$. Call $|x_1\rangle = P|x\rangle$ and $|x_2\rangle = [|x\rangle - P|x\rangle]$.

$$P|x\rangle = P|x_1\rangle + P|x_2\rangle = P^2|x\rangle + P|x_2\rangle = P|x\rangle + P|x_2\rangle \Rightarrow P|x_2\rangle = |0\rangle \Rightarrow |x_2\rangle\epsilon N(P)$$
$$\Rightarrow P|x\rangle = P|x_1\rangle + P|x_2\rangle = P|x_1\rangle \Rightarrow P|x_1\rangle = |x_1\rangle \Rightarrow |x_1\rangle\epsilon\{|y\rangle\epsilon V : P|y\rangle = |y\rangle\}$$

Notice that: $N(P) \cap \{|y\rangle\epsilon V : P|y\rangle = |y\rangle\}$ and that $V = \{|y\rangle\epsilon V : P|y\rangle = |y\rangle\} + N(P)$.

Hence, $V = \{|y\rangle\epsilon V : P|y\rangle = |y\rangle\} \oplus N(P)$ and P is a projection on $\{|y\rangle\epsilon V : P|y\rangle = |y\rangle\}$ along $N(P)$.

Proof of a projection P is orthogonal iff P has an adjoint $P^{\dagger}$ and $P^2 = P = P^{\dagger}$:
(i) P is a projection $\Leftrightarrow P^2 = P$
(ii) P orthogonal $\Rightarrow P$ has an adjoint $P^{\dagger}$ and $P = P^{\dagger}$:

$R(P)^{\perp} = N(P)$ implies: $|x_1\rangle \epsilon R(P), |y_2\rangle \epsilon N(P) \Rightarrow \langle y_2|x_1\rangle = \langle x_1|y_2\rangle = 0$

For any $|x\rangle, |y\rangle \epsilon V$, then we can write $|x\rangle = |x_1\rangle + |x_2\rangle$ and $|y\rangle = |y_1\rangle + |y_2\rangle$ such that $|x_1\rangle, |y_1\rangle \epsilon R(P)$ and $|x_2\rangle, |y_2\rangle \epsilon N(P)$

$\langle x|P|y\rangle = [\langle x_1| + \langle x_2|]P|y\rangle = [\langle x_1| + \langle x_2|]|y_1\rangle = \langle x_1|y_1\rangle = \langle x_1|y\rangle = [\langle x|P^{\dagger}]|y\rangle = \langle x|P^{\dagger}|y\rangle$

Since $|x\rangle$ and $|y\rangle$ are arbitrary, $P^{\dagger}$ exists and $P = P^{\dagger}$

(iii) P has an adjoint $P^{\dagger}$ and $P^2 = P = P^{\dagger} \Rightarrow P$ orthogonal:

(a) Let $|x\rangle \epsilon R(P)$ and $|y\rangle \epsilon N(P)$.

$\langle x|y\rangle = [\langle x|P^{\dagger}]|y\rangle = \langle x|P^{\dagger}|y\rangle = \langle x|P|y\rangle = \langle x|(P|y\rangle) = \langle x|0\rangle = 0$

$\Rightarrow |x\rangle \epsilon N(P)^{\perp}$ and $|y\rangle \epsilon R(P)^{\perp} \Rightarrow R(P) \subset N(P)^{\perp}$ and $N(P) \subset R(P)^{\perp}$ (since $|x\rangle$ and $|y\rangle$ were arbitrary)

(b) Consider $|z\rangle \epsilon N(P)^{\perp}$ and $|w\rangle = [|z\rangle - P|z\rangle] \epsilon N(P)$

$$\langle w|w\rangle = \langle z|w\rangle - [\langle z|P^{\dagger}]|w\rangle = \langle z|w\rangle - \langle z|P^{\dagger}|w\rangle = \langle z|w\rangle - \langle z|P|w\rangle = \langle z|w\rangle - \langle z|[P|z\rangle - P^2|z\rangle]$$

$$= \langle z|w\rangle - \langle z|[P|z\rangle - P|z\rangle] = \langle z|w\rangle - \langle z|0\rangle = \langle z|w\rangle = 0$$

Hence, $|w\rangle = |0\rangle$, which implies $|z\rangle = P|z\rangle \Rightarrow |z\rangle \epsilon R(P) \Rightarrow N(P)^{\perp} \subset R(P)$

(c) Consider $|z\rangle \epsilon R(P)^{\perp}$ and any $|x\rangle = [|x_1\rangle + |x_2\rangle] \epsilon V$

$[\langle z|P^{\dagger}]|x\rangle = \langle z|P^{\dagger}|x\rangle = \langle z|P|x\rangle = \langle z|x_1\rangle = 0$

Hence, $P|z\rangle = |0\rangle \Rightarrow |z\rangle \epsilon N(P) \Rightarrow R(P)^{\perp} \subset N(P)$

(d) $R(P) \subset N(P)^{\perp}$ and $N(P) \subset R(P)^{\perp}$ and $N(P)^{\perp} \subset R(P)$ and $R(P)^{\perp} \subset N(P) \Rightarrow R(P) = N(P)^{\perp}$ and $N(P) = R(P)^{\perp}$

Hence, P is orthogonal.