The Forced Harmonic Oscillator, Decoherence, and Schrödinger Cat States

The Forced Harmonic Oscillator

The harmonic oscillator is a very significant problem in quantum mechanics, because it is one of the very few, relatively non-trivial problems that has an exact analytic solution. In addition to this, the harmonic oscillator solution and algebraic formalism has applications throughout modern physics (e.g. many particle systems, QED, quantum field theory, approximations of real systems such as molecular bonds, etc.). As it turns out, adding a force term to the linear harmonic oscillator gives a quantum system that can also be solved for exactly. This is of great value because it means we can produce analytic solutions to systems that are not quite as idealized as the simple LHO and it allows us to know exactly how harmonic oscillator states, particularly coherent states, evolve in the presence of force. Although the first exact solution of the Schrödinger equation for the forced quantum harmonic oscillator was found by Husimi in 1953 (and independently by Kerner in 1957), it turns out that Caltech's very own Richard Feynman was actually the first to treat the FHO problem in 1948, using the path integral method (of course). These first treatments of the FHO were done before the first comprehensive treatment of coherent states was given by Glauber in 1963, and consequently the early literature on the FHO problem utilizes a somewhat cumbersome treatment of the problem (i.e. solving the differential equations for displaced gaussians and Hermite polynomials with moving centers, or, in Feynman's case, the solution being given in a largely unfamiliar formulation of quantum mechanics). The FHO has been solved for several different ways and for several different levels of generality (i.e. inclusion of damping, allowance for time-dependent frequency and/or mass, use of path integral method, use of the Heisenberg picture with invarients of motion, etc.), but I will present the constant mass, constant frequency, no damping problem and use the notation and techniques of the LHO and interaction picture that we learned in in class so as to make the lecture more accessible.

Consider the Hamiltonian of a harmonic oscillator system that also has an interaction potential $\mathbf{V}(t)$:

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{V}(t) = \hbar\omega\left(\mathbf{N} + \frac{1}{2}\right) + \mathbf{V}(t)$$

where $\mathbf{H}_0 = \hbar\omega\left(\mathbf{N}+\frac{1}{2}\right)$ is the Hamitonian of the simple LHO. The first thing to do is convert to the Interaction picture where $\left|\widetilde{\Psi}(0)\right\rangle = |\Psi(0)\rangle$:

$$\left|\widetilde{\Psi}(t)\right\rangle = \exp(\frac{i}{\hbar}\mathbf{H}_0 t)|\Psi(t)\rangle$$
$$\widetilde{\mathbf{L}} = \exp(\frac{i}{\hbar}\mathbf{H}_0 t)\mathbf{L}\exp(-\frac{i}{\hbar}\mathbf{H}_0 t)$$
$$i\hbar\frac{d}{dt}\left|\widetilde{\Psi}(t)\right\rangle = \widetilde{\mathbf{V}}(t)\left|\widetilde{\Psi}(t)\right\rangle$$
$$\widetilde{\mathbf{T}}(t_2,t_1) = \exp(\frac{i}{\hbar}\mathbf{H}_0 t_2)\mathbf{T}(t_2,t_1)\exp(-\frac{i}{\hbar}\mathbf{H}_0 t_1)$$
$$i\hbar\frac{d}{dt}\widetilde{\mathbf{T}}(t,t_0) = \widetilde{\mathbf{V}}(t)\widetilde{\mathbf{T}}(t,t_0)$$

Recall that the solution for the time development operator is:

$$\begin{aligned}\widetilde{\mathbf{T}}(t,t_0) &= \mathbf{1} - \frac{i}{\hbar}\int_{t_0}^{t}\widetilde{\mathbf{V}}(t')dt' + \left(-\frac{i}{\hbar}\right)^2\int_{t_0}^{t}\widetilde{\mathbf{V}}(t')dt'\int_{t_0}^{t'}\widetilde{\mathbf{V}}(t'')dt'' + \ldots \\ &= \mathbf{1} + \sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n\int_{t_0}^{t}dt_1\ldots\int_{t_0}^{t}dt_n T\left\{\widetilde{\mathbf{V}}(t_1)\ldots\widetilde{\mathbf{V}}(t_n)\right\} \\ &= T\left\{\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\widetilde{\mathbf{V}}(t')dt'\right]\right\}\end{aligned}$$

In order to get the time development operator in a more managable form before addressing the FHO problem, we can consider a case of $\mathbf{V}(t)$ with additional general properties. We will see below that commutation of $\widetilde{\mathbf{V}}(t)$ at different times it too strong a condition for the FHO problem, but commutation with the commutator (i.e. $\left[\widetilde{\mathbf{V}}(t),\left[\widetilde{\mathbf{V}}(t'),\widetilde{\mathbf{V}}(t'')\right]\right] = 0$) is exactly the property exhibited by the FHO, and indeed allows us to assume a nicer form for the time development operator before plugging in the forcing potential. We begin by breaking up the integral into the sum of integrals over infinitesimal lengths of time $\epsilon = \frac{t-t_0}{N}$ with $N \to \infty$, defining:

$$\mathbf{V}_n = -\frac{i}{\hbar}\int_{t_0+(n-1)\epsilon}^{t_0+n\epsilon}\widetilde{\mathbf{V}}(t')dt'$$

Noting that even if the individual $\mathbf{V}_n$'s do not commute, their time intervals are infinitesimal and hence are not subject to internal time ordering, we can write the time development operator as:

$$\begin{aligned}\widetilde{\mathbf{T}}(t,t_0) &= \widetilde{\mathbf{T}}(t,t-\epsilon)\widetilde{\mathbf{T}}(t-\epsilon,t-2\epsilon)\ldots\widetilde{\mathbf{T}}(t_0+2\epsilon,t_0+\epsilon)\widetilde{\mathbf{T}}(t_0+\epsilon,t_0) \\ &= \lim_{N\to\infty} e^{\mathbf{V}_N}e^{\mathbf{V}_{N-1}}\ldots e^{\mathbf{V}_2}e^{\mathbf{V}_1}\end{aligned}$$

Then assuming $\left[\widetilde{\mathbf{V}}(t),\left[\widetilde{\mathbf{V}}(t'),\widetilde{\mathbf{V}}(t'')\right]\right] = 0$, we can then apply the Campbell-Baker-Hausdorff relation:

$$e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{A}+\mathbf{B}+[\mathbf{A},\mathbf{B}]/2}$$

which holds when **A** and **B** both commute with [**A**, **B**], and then carry out the limit $N \to \infty$ to get:

$$\begin{aligned}\widetilde{\mathbf{T}}(t,t_0) &= \lim_{N\to\infty}\exp\left[\sum_{n=1}^{N}\left(\mathbf{V}_n + \frac{1}{2}\left[\mathbf{V}_n, \sum_{k=1}^{n}\mathbf{V}_k\right]\right)\right] \\ &= \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\widetilde{\mathbf{V}}(t')dt' - \frac{1}{2\hbar^2}\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\left[\widetilde{\mathbf{V}}(t'),\widetilde{\mathbf{V}}(t'')\right]\right]\end{aligned}$$

Now we have reached the point where we want to plug in an actual interaction potential.

On a side note, we can now add to our list of special cases of solutions of the time evolution operator: When the Hamiltonian commutes with its commutator (i.e. $\mathbf{H}(t)$ commutes with $[\mathbf{H}(t'),\mathbf{H}(t'')]$ for any times: t,t',t'') then we can write the evolution operator as:

$$\mathbf{T}(t,t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\mathbf{H}(t')dt' - \frac{1}{2\hbar^2}\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''[\mathbf{H}(t'),\mathbf{H}(t'')]\right]$$

which is particularly nice when the commutator is a c-number. Notice this reduces to the familiar solution when the Hamiltonian commutes with itself at different times (i.e. when the commutator is zero).

Now we consider a spatially uniform time-dependent force $F(t)$, which corresponds to a potential in the Schrödinger picture given by:

$$\mathbf{V}(t) = -\mathbf{x}F(t) = f(t)[\mathbf{a}+\mathbf{a}^\dagger]$$

$$f(t) = -\sqrt{\frac{\hbar}{2m\omega}}\,F(t)$$

In the interaction picture this becomes:

$$\begin{aligned}\widetilde{\mathbf{V}}(t) &= \exp(\tfrac{i}{\hbar}\mathbf{H}_0 t)f(t)(\mathbf{a}+\mathbf{a}^\dagger)\exp(-\tfrac{i}{\hbar}\mathbf{H}_0 t)\\ &= \exp(i\omega\mathbf{a}^\dagger\mathbf{a}t)f(t)(\mathbf{a}+\mathbf{a}^\dagger)\exp(-i\omega\mathbf{a}^\dagger\mathbf{a}t)\\ &= f(t)(\mathbf{a}e^{-i\omega t}+\mathbf{a}^\dagger e^{i\omega t})\end{aligned}$$

giving the unequal time commutation of the potential:

$$\begin{aligned}\left[\widetilde{\mathbf{V}}(t'),\widetilde{\mathbf{V}}(t'')\right] &= f(t')f(t'')\left[\mathbf{a}e^{-i\omega t'}+\mathbf{a}^\dagger e^{i\omega t'},\mathbf{a}e^{-i\omega t''}+\mathbf{a}^\dagger e^{i\omega t''}\right]\\ &= f(t')f(t'')\left(e^{-i\omega(t'-t'')}-e^{i\omega(t'-t'')}\right)\\ &= -i2f(t')f(t'')\sin(\omega(t'-t''))\end{aligned}$$

which is a c-number (and so commutes with $\widetilde{\mathbf{V}}(t)$ for all times as I claimed it would). We can then define the terms:

$$\zeta(t,t_0) = -\frac{i}{\hbar}\int_{t_0}^{t}f(t')e^{i\omega t'}dt'$$

$$\beta(t,t_0) = \frac{1}{\hbar^2}\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''f(t')f(t'')\sin(\omega(t'-t''))$$

and then plugging these back into the expression for the interaction picture time development operator, we get:

$$\begin{aligned}\widetilde{\mathbf{T}}(t,t_0) &= \exp(\zeta(t,t_0)\mathbf{a}^\dagger-\zeta^*(t,t_0)\mathbf{a}+i\beta(t,t_0))\\ &= e^{i\beta(t,t_0)}\exp(\zeta(t,t_0)\mathbf{a}^\dagger-\zeta^*(t,t_0)\mathbf{a})\\ &= e^{i\beta(t,t_0)}\mathbf{D}(\zeta(t,t_0))\end{aligned}$$

Now we can convert back to the Schrödinger picture:

$$
\begin{aligned}
\mathbf{T}(t,t_0) &= \exp(-\tfrac{i}{\hbar}\mathbf{H}_0 t)\widetilde{\mathbf{T}}(t,t_0)\exp(\tfrac{i}{\hbar}\mathbf{H}_0 t_0) \\
&= \exp(-\tfrac{i}{\hbar}\mathbf{H}_0 t)e^{i\beta(t,t_0)}\mathbf{D}(\zeta(t,t_0))\exp(\tfrac{i}{\hbar}\mathbf{H}_0 t_0) \\
&= e^{i\beta(t,t_0)}\mathbf{D}(\zeta(t,t_0)e^{-i\omega t})\exp\left(-\tfrac{i}{\hbar}\mathbf{H}_0 t\right)\exp\left(\tfrac{i}{\hbar}\mathbf{H}_0 t_0\right) \\
&= e^{i\beta(t,t_0)}\mathbf{D}(\zeta(t,t_0)e^{-i\omega t})\exp\left(-\tfrac{i}{\hbar}\mathbf{H}_0 (t-t_0)\right) \\
&= e^{i\beta(t,t_0)}\mathbf{D}(\zeta(t,t_0)e^{-i\omega t})\mathbf{T}_0(t,t_0)
\end{aligned}
$$

where $\mathbf{T}_0(t,t_0)$ is the time development operator for the LHO (Hamiltonian $\mathbf{H}_0$). Thus we have solved for the time evolution of the forced harmonic oscillator system. What we have found for the solutions of the FHO are states of the LHO that are displaced and have an overall change of phase. Here is a collection of the important terms:

$$
\begin{aligned}
\mathbf{T}(t,t_0) &= e^{i\beta(t,t_0)}\mathbf{D}(\zeta(t,t_0)e^{-i\omega t})\mathbf{T}_0(t,t_0) \\
\widetilde{\mathbf{T}}(t,t_0) &= e^{i\beta(t,t_0)}\mathbf{D}(\zeta(t,t_0)) \\
\zeta(t,t_0) &= -\frac{i}{\hbar}\int_{t_0}^{t} f(t')e^{i\omega t'}dt' \\
\beta(t,t_0) &= \frac{1}{\hbar^2}\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt'' f(t')f(t'')\sin(\omega(t'-t'')) \\
f(t) &= -\sqrt{\frac{\hbar}{2m\omega}}\,F(t) \\
\mathbf{V}(t) &= -\mathbf{x}F(t) \\
\mathbf{H} &= \mathbf{H}_0 + \mathbf{V}(t) = \hbar\omega\left(\mathbf{N} + \frac{1}{2}\right) + \mathbf{V}(t)
\end{aligned}
$$

The analytic solution of the FHO obtained by explicitly solving the Schrödinger wave equation gives essentially the same result in the following form:

$$
\psi(x,t) = \chi(x-\xi(t),t)\exp\left[\frac{im}{\hbar}\frac{d\xi(t)}{dt}(x-\xi(t)) + \frac{i}{\hbar}\int_0^t L dt'\right]
$$

where $\chi(x,t)$ satisfies the wave equation for the unforced LHO and can be taken to be either a stationary solution (number state) or nonstationary state (coherent state), the center of the wave packet $\xi(t)$ satisfies the classical equation of motion of a forced harmonic oscillator: $\frac{d^2\xi}{dt^2} + \omega^2\xi = \frac{F(t)}{m}$, and L is the classical Lagrangian: $L = \frac{1}{2}m\left(\frac{d\xi}{dt}\right)^2 - \frac{1}{2}m\omega^2\xi^2 + F(t)\xi$

Here is a list of several treatments of the forced harmonic oscillator:

R. P. Feynman, "Space-time approach to non-relativistic quantum mechanics," *Rev. Mod. Phys.* **20**, 367-387 (1948).

R. P. Feynman, "Mathematical formulation of the quantum theory of electromagnetic interaction," Phys. *Rev.* **80**, 440-457 (1950).

Solved for using the path integral method. Feynman used the FHO propagater in application to QED.

K. Husimi, "Miscellanea in Elementary Quantum Mechanics, II," *Progr. Theor. Phys.* **9**,

381-402 (1953).

Solved for time-dependent frequency. Uses gaussian and Hermite polynomial method. (A bit archaic, but not too hard to read.)

E. H. Kerner, “Note On The Forced And Damped Oscillator In Quantum Mechanics,” *Can. J. Phys.* **36**, 371-377 (1957).

Solved the forced and/or damped oscillator, finding that damping gives a contraction of the gaussian wave function into a delta function at the oscillating classical position.

E. Merzbacher, *Quantum Mechanics*, (Wiley, New York, 1998), 3rd ed.

Merzbacher presents two ways of solving the FHO in Chapter 14.6. The first, using Green’s functions in the Heisenberg picture, is particularly slick. However, I tried to somewhat follow his second approach in my above presentation (to give the reader an easily accessed reference), since it is ideal for using the displacement operator notation, and thus more convenient for dealing with coherent states.

H. Kim, M. Lee, J. Ji, and J. K. Kim, “Heisenberg-picture approach to the exact quantum motion of a time-dependent forced harmonic oscillator,” *Phys. Rev. A* **53**, (1996).

Solved for time-dependent frequency and mass with the inclusion of damping, using generalized invarients in the Heisenberg picture. (Very technical.)

Coherent States in the FHO

To begin with, here are some properties of the coherent states for the LHO that were not explicitly covered in lecture notes, but may be useful later.

$$
\begin{aligned}
|\alpha\rangle &= \mathbf{D}(\alpha)|0\rangle = \exp(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a})|0\rangle = \exp(-|\alpha|^2/2)\exp(\alpha\mathbf{a}^\dagger)|0\rangle \\
&= \sum_{n=0}^{\infty}|n\rangle\langle n|\exp(-|\alpha|^2/2)\exp(\alpha\mathbf{a}^\dagger)|0\rangle \\
&= \sum_{n=0}^{\infty}|n\rangle\langle n|\exp(-|\alpha|^2/2)\sum_{k=0}^{\infty}\frac{(\alpha\mathbf{a}^\dagger)^k}{k!}|0\rangle \\
&= \sum_{n=0}^{\infty}\exp(-|\alpha|^2/2)\frac{\alpha^n}{\sqrt{n!}}|n\rangle
\end{aligned}
$$

Hence, a coherent state is a Poisson distribution of number states:

$$P_n(\alpha) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$$

with number operator expectation value:

$$\langle\mathbf{N}\rangle = \langle\mathbf{a}^\dagger\mathbf{a}\rangle = \langle\alpha|\mathbf{a}^\dagger\mathbf{a}|\alpha\rangle = \langle\alpha|\alpha^*\alpha|\alpha\rangle = \alpha^*\alpha\langle\alpha|\alpha\rangle = \alpha^*\alpha = |\alpha|^2$$

where I used the fact that coherent states are eigenstates of **a**. This gives the energy expectation value:

$$\langle\mathbf{H}\rangle = \left\langle\hbar\omega\left(\mathbf{N}+\frac{1}{2}\right)\right\rangle = \hbar\omega\left(|\alpha|^2+\frac{1}{2}\right)$$

Now we examine the behavior of coherent states in the (uniformly) forced harmonic oscillator. Consider the Interaction picture evolution of the initial state $|\Psi(0)\rangle = \left|\widetilde{\Psi}(0)\right\rangle = |\alpha\rangle$
:

$$
\begin{aligned}
\left|\widetilde{\Psi}(t)\right\rangle &= \widetilde{\mathbf{T}}(t,0)\left|\widetilde{\Psi}(0)\right\rangle = \widetilde{\mathbf{T}}(t,0)|\alpha\rangle = \widetilde{\mathbf{T}}(t,0)\mathbf{D}(\alpha)|0\rangle \\
&= e^{i\beta(t,0)}\mathbf{D}(\zeta(t,0))\mathbf{D}(\alpha)|0\rangle \\
&= e^{i\beta(t)}\exp(\zeta(t)\mathbf{a}^\dagger - \zeta^*(t)\mathbf{a})\exp(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a})|0\rangle \\
&= e^{i\beta(t)}\exp(\zeta(t)\alpha^* - \zeta^*(t)\alpha)\exp((\alpha+\zeta(t))\mathbf{a}^\dagger - (\alpha^*+\zeta^*(t))\mathbf{a})|0\rangle \\
&= e^{i\beta(t)}\exp\left[\frac{1}{2}(\zeta(t)\alpha^* - \zeta^*(t)\alpha)\right]\mathbf{D}(\alpha+\zeta(t))|0\rangle \\
&= e^{i\beta(t)}\exp\left[\frac{1}{2}(\zeta(t)\alpha^* - \zeta^*(t)\alpha)\right]|\alpha+\zeta(t)\rangle \\
&= e^{i\gamma(t)}|\alpha+\zeta(t)\rangle
\end{aligned}
$$

where we have defined $\beta(t) = \beta(t,0)$ and $\zeta(t) = \zeta(t,0)$, and the real phase $\gamma(t)$ is:

$$
\begin{aligned}
\gamma(t) &= \beta(t) - \frac{i}{2}[\zeta(t)\alpha^* - \zeta^*(t)\alpha] \\
&= \beta(t) - \frac{1}{2}\left[\frac{\alpha^*}{\hbar}\int_0^t f(t')e^{i\omega t'}dt' + \frac{\alpha}{\hbar}\int_0^t f(t')e^{-i\omega t'}dt'\right] \\
&= \beta(t) - \left[\frac{\mathrm{Re}(\alpha)}{\hbar}\int_0^t f(t')\cos(\omega t')dt' + \frac{\mathrm{Im}(\alpha)}{\hbar}\int_0^t f(t')\sin(\omega t')dt'\right] \\
&= -\frac{\mathrm{Re}(\alpha)}{\hbar}\int_0^t f(t')\cos(\omega t')dt' - \frac{\mathrm{Im}(\alpha)}{\hbar}\int_0^t f(t')\sin(\omega t')dt' + \frac{1}{\hbar^2}\int_0^t dt'\int_0^{t'} dt'' f(t')f(t'')\sin(\omega(t'-t''))
\end{aligned}
$$

giving the resulting time evolved state:

$$
\begin{aligned}
|\Psi(t)\rangle &= \exp(-\frac{i}{\hbar}\mathbf{H}_0 t)\left|\widetilde{\Psi}(t)\right\rangle \\
&= \exp(-\frac{i}{\hbar}\mathbf{H}_0 t)e^{i\gamma(t)}|\alpha+\zeta(t)\rangle \\
&= e^{i\gamma(t)}e^{-\frac{i}{2}\omega t}|(\alpha+\zeta(t))e^{-i\omega t}\rangle
\end{aligned}
$$

Thus, what we have discovered is the property that coherent states in the FHO evolve into other coherent states (under spatially uniform forcing). The immediate consequence of this is that we now understand how to physically create coherent states: reduce (by cooling or whatever) harmonic oscillators to their ground state and then apply a (spatially uniform) force pulse. Fortunately, this method of creating coherent states does not stray far from our intuition.

Decoherence of a Superposition of Coherent States

Consider the initial state of an equal superposition of two arbitrary coherent states with arbitrary relative phases:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(e^{i\theta_1(0)}|\alpha_1(0)\rangle + e^{i\theta_2(0)}|\alpha_2(0)\rangle)$$

This gives the forced time evolved state:

$$
\begin{aligned}
|\Psi(t)\rangle &= \frac{1}{\sqrt{2}}\left(e^{i\theta_1(t)}|\alpha_1(t)\rangle + e^{i\theta_2(t)}|\alpha_2(t)\rangle\right) \\
\theta_1(t) &= \theta_1(0) + \gamma_1(t) - \frac{\omega t}{2} \\
\theta_2(t) &= \theta_2(0) + \gamma_2(t) - \frac{\omega t}{2} \\
\alpha_1(t) &= (\alpha_1(0) + \zeta(t))e^{-i\omega t} \\
\alpha_2(t) &= (\alpha_2(0) + \zeta(t))e^{-i\omega t}
\end{aligned}
$$

Decoherence of this superposition of states is due to changes in phase difference, so we define the change in phase difference:

$$
\begin{aligned}
\Delta\theta(t) &= [\theta_2(t) - \theta_1(t)] - [\theta_2(0) - \theta_1(0)] \\
&= \gamma_2(t) - \gamma_1(t) \\
&= -\frac{1}{2}\left[\left(\frac{\alpha_2^*(0) - \alpha_1^*(0)}{\hbar}\right)\int_0^t f(t')e^{i\omega t'}dt' + \left(\frac{\alpha_2(0) - \alpha_1(0)}{\hbar}\right)\int_0^t f(t')e^{-i\omega t'}dt'\right] \\
&= -\frac{1}{2}\left[\frac{\Delta\alpha^*}{\hbar}\int_0^t f(t')e^{i\omega t'}dt' + \frac{\Delta\alpha}{\hbar}\int_0^t f(t')e^{-i\omega t'}dt'\right]
\end{aligned}
$$

for the initial amplitude difference $\Delta\alpha = \Delta\alpha(0) = \alpha_2(0) - \alpha_1(0)$. Note that the quantity $|\Delta\alpha(t)| = |\alpha_2(t) - \alpha_1(t)|$ remains constant under FHO time evolution. Taking the statisical mechanical ensemble average of the square of the change in phase difference, we get:

$$
\begin{aligned}
\langle(\Delta\theta(t))^2\rangle &= \frac{1}{4\hbar^2}\left\langle \begin{matrix} (\Delta\alpha^*)^2\int_0^t dt'\int_0^t dt'' f(t')f(t'')e^{i\omega(t'+t'')} \\ +2|\Delta\alpha|^2\int_0^t dt'\int_0^t dt'' f(t')f(t'')e^{i\omega(t'-t'')} \\ +(\Delta\alpha)^2\int_0^t dt'\int_0^t dt'' f(t')f(t'')e^{-i\omega(t'+t'')} \end{matrix}\right\rangle \\
&= \frac{1}{4\hbar^2}\left[\begin{matrix} (\Delta\alpha^*)^2\int_0^t dt'\int_0^t dt'' \langle f(t')f(t'')\rangle e^{i\omega(t'+t'')} \\ +2|\Delta\alpha|^2\int_0^t dt'\int_0^t dt'' \langle f(t')f(t'')\rangle e^{i\omega(t'-t'')} \\ +(\Delta\alpha)^2\int_0^t dt'\int_0^t dt'' \langle f(t')f(t'')\rangle e^{-i\omega(t'+t'')} \end{matrix}\right]
\end{aligned}
$$

At this point, we need to determine an explicit value for the force $f(t)$ that causes decoherence. This decoherence force is a property of the particular physical system in question, and can often be described using the theory of random processes. Details on the theory of random processes can be found in most statistical mechanics texts, such as Reif or Pathria, or Blanford and Thorne’s Applications of Classical Physics text Chapter 5 (currently in the form of lecture notes on the Phys 136 course web-page). The results that we need to use for a stationary random variable $f(t)$ with the autocorrelation function $C_f(\tau) = \langle f(t)f(t+\tau)\rangle$ and the spectral density (power spectrum) $S_f(\nu)$ for positive frequency ν are:

$$
\begin{aligned}
S_f(\nu) &= 4\int_0^\infty C_f(\tau)\cos(2\pi\nu\tau)d\tau \\
C_f(\tau) &= \int_0^\infty S_f(\nu)\cos(2\pi\nu\tau)d\nu
\end{aligned}
$$

Since we do not have a specified system that we know the explicit details of $f(t)$, we will look at the case of a ”white noise” power spectrum (i.e. $S_f(\nu) = const.$) which gives $C_f(\tau) = c\delta(\tau)$, for some constant c. This may correspond to the effects of coupling to a thermal reservoir, the presence of a uniform electric field for a charged particle, vacuum flucations, etc. From this power spectrum, we get the result:

$$\langle(\Delta\theta(t))^2\rangle = \frac{c}{4\hbar^2}\left[(\Delta\alpha^*)^2\int_0^t dt' e^{i2\omega t'} + 2|\Delta\alpha|^2\int_0^t dt' + (\Delta\alpha)^2\int_0^t dt' e^{-i2\omega t'}\right]$$

It is clear that the first and third integrals oscillate (giving bounded values), while the second integral is simply equal to t, giving the large t approximation ($\omega t \gg 1$):

$$\langle(\Delta\theta(t))^2\rangle \approx \frac{c|\Delta\alpha|^2}{2\hbar^2}t$$

Hence, we get the decoherence time scale (the time it takes for about 1 radian of increase in the rms change in phase difference):

$$\tau_{decoherence} = \frac{2\hbar^2}{c|\Delta\alpha|^2}$$

This implies that the larger the initial separation of coherent state amplitudes, the shorter the decoherence time. Applying the same technique to the changes in amplitude, we get:

$$\begin{aligned}\left\langle|\alpha_j(t)|^2 - |\alpha_j(0)|^2\right\rangle &= \left\langle|\zeta(t)|^2\right\rangle \\ &= \frac{1}{\hbar^2}\int_0^t dt'\int_0^t dt''\langle f(t')f(t'')\rangle e^{i\omega(t'-t'')} \\ &= \frac{c}{\hbar^2}\int_0^t dt' = \frac{c}{\hbar^2}t\end{aligned}$$

which can be used to get the mean fractional change in energy of the single coherent states:

$$\frac{\langle\Delta E_j(t)\rangle}{E_j(0)} = \frac{\left\langle|\alpha_j(t)|^2 - |\alpha_j(0)|^2\right\rangle}{|\alpha_j(0)|^2 + \frac{1}{2}} = \frac{c}{\hbar^2\left(|\alpha_j(0)|^2 + \frac{1}{2}\right)}t$$

so that in the decoherence time, this gives:

$$\frac{\langle\Delta E_j(\tau_d)\rangle}{E_j(0)} = \frac{2}{|\Delta\alpha|^2\left(|\alpha_j(0)|^2 + \frac{1}{2}\right)}$$

which is small for large initial coherent state amplitude $|\alpha_j(0)|^2$ and for large initial separations $|\Delta\alpha|^2$.

“Schrödinger Cat” States

I assume we are all familiar with Schrödinger’s thought experiment of a cat in the death box that would put the the cat in a superposition of alive and dead states. Schrödinger’s dilema was how to reconcile quantum mechanics allowing for such superpositions of two macroscopically distinct states with the readily observable macroscopic world in which these superpositions do not appear. Since we know creating a Schrödinger Cat superposition with strictly macroscopic systems is unrealistic, we will appeal to mesoscopic systems, which are systems with both macroscopic and microscopic features. By no accident, coherent states of the quantum harmonic oscillator are precisely such mesoscopic states.

They are clearly microscopic in that they are quantum mechanical solutions, yet they possess the macroscopic property of having their wave packet centered around a classically oscillating position. As the quantity α represents the amplitude of this classical motion for the coherent state $|\alpha\rangle$, it is also a sort of measure of how "macroscopic-like" the state is (i.e. larger α means more macroscopic-like). Taking the superposition of two coherent states:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(e^{i\theta_1(0)}|\alpha_1(0)\rangle + e^{i\theta_2(0)}|\alpha_2(0)\rangle)$$

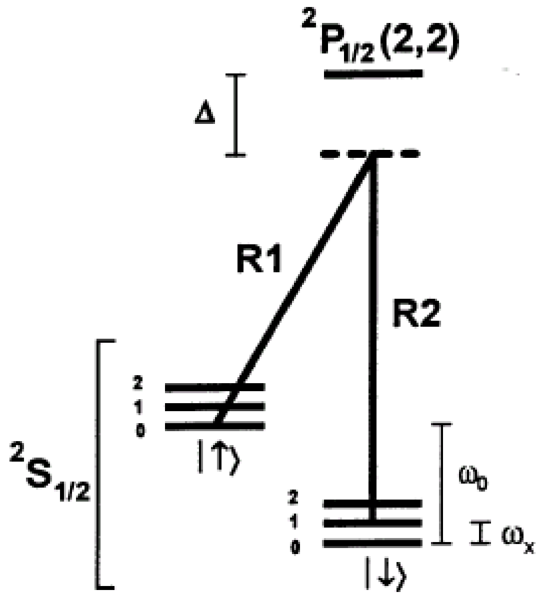
we realize that in order to have the states $|\alpha_1(0)\rangle$ and $|\alpha_2(0)\rangle$ act like two distinct macroscopic states, we require their individual amplitudes $\alpha_1(0)$ and $\alpha_2(0)$ as well as their separation $\Delta\alpha = \alpha_2(0) - \alpha_1(0)$ to be large. (Recall that the inner product of two coherent states is $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$ so that increasing $\Delta\alpha$ makes the states more distinguishable from one another.) So letting our Schrödinger Cat state be the above superposition of two coherent states with $\alpha_1(0), \alpha_2(0),$ and $\Delta\alpha$ large, we realize that the results of the last section apply directly to this state. The results tell us that $\Delta\alpha$ large makes the decoherence time small, and $\alpha_1(0), \alpha_2(0),$ and $\Delta\alpha$ large indicates that decoherence occurs much more rapidly that changes in energy. This decoherence effect demonstrates for us why we do not see quantum superpositions of classical states in the macroscopic world, and gives a good indication of how difficult it can be to maintain such superpositions even for mesoscopic states.

The term "Schrödinger Cat state" is not universally agreed upon to mean the superposition of two macroscopic states as assumed above. An alternate definition of a Schrödinger Cat state is an entangled state of two macroscopic states with two corresponding purely quantum states (i.e. spin), such as the following state:

$$|\Psi\rangle = \frac{|\alpha\rangle \otimes |\uparrow\rangle + |-\alpha\rangle \otimes |\downarrow\rangle}{\sqrt{2}}$$

These Schrödinger Cat states have essentially the same decoherence effect as the alternate definition, so there is no need to re-examine this effect.

Now let's take a look at how these Schrödinger Cat states have actually been realized experimentally. I will present the method used by C. Monroe, et. al to create these states. A single $^9Be^+$ ion is confined in a coaxial-resonator radio frequency (RF)-ion trap. The harmonic oscillator states are provided by the motion of the ion in the trap and the internal spin states are provided by the stable hyperfine ground states: $|\uparrow\rangle = {}^2S_{1/2}(F{=}1, m_f = -1)$ and $|\downarrow\rangle = {}^2S_{1/2}(F{=}2, m_f = -2)$. Transitions between $|\uparrow\rangle$ and $|\downarrow\rangle$ are obtained by a two Raman beam "carrier pulse" through an intermediate detuned excited state. These transitions do not significantly affect the oscillator states, because the beams are copropagating. By adjusting the exposure time of the carrier beams, one can generate different effects, in particular a π-pulse (1/2 Rabi cycle) flips the internal states and a $\frac{\pi}{2}$-pulse (1/4 Rabi cycle) splits both internal states. Coherent states are generated (from the ground state $|0\rangle$) for the $|\uparrow\rangle$ state only, by applying a force pulse from a displacement beam that is polarized such that it does not couple to the $|\downarrow\rangle$ state, and hence leaves it unaffected.



Given the ability to perform the manipulations described above on this trapped ion system, the following steps will generate the Schrödinger Cat state: (a) generate an initial wave packet in the ground state $|0\rangle \otimes |\downarrow\rangle$ by laser-cooling the ion, (b) split the wave packet by a $\frac{\pi}{2}$-pulse on the carrier, (c) excite the $|0\rangle \otimes |\uparrow\rangle$ wave packet to the coherent state $|\alpha\rangle \otimes |\uparrow\rangle$ by applying the force $F_1 = F$ displacement beam, (d) interchange the spin states with a π-pulse on the carrier, (e) apply the force $F_2 = -F$ displacement beam to excite the new $|0\rangle \otimes |\uparrow\rangle$ wave packet to the coherent state $|-\alpha\rangle \otimes |\uparrow\rangle$ and the result is our desired Schrödinger Cat state. Here are the states at each step:

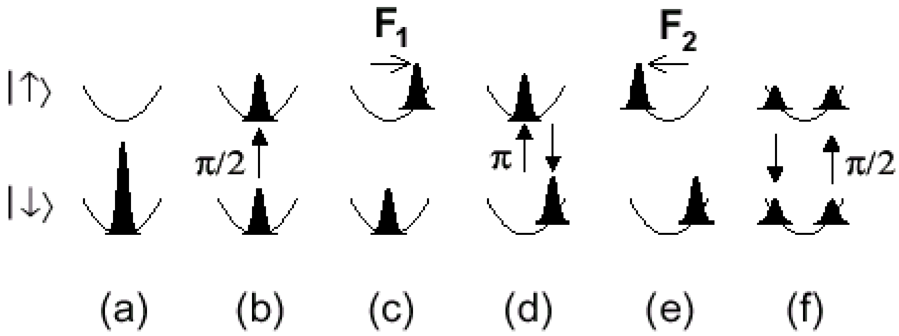
$$|\Psi_{(a)}\rangle = |0\rangle \otimes |\downarrow\rangle$$

$$|\Psi_{(b)}\rangle = \frac{|0\rangle \otimes |\uparrow\rangle + |0\rangle \otimes |\downarrow\rangle}{\sqrt{2}}$$

$$|\Psi_{(c)}\rangle = \frac{|\alpha\rangle \otimes |\uparrow\rangle + |0\rangle \otimes |\downarrow\rangle}{\sqrt{2}}$$

$$|\Psi_{(d)}\rangle = \frac{|0\rangle \otimes |\uparrow\rangle + |\alpha\rangle \otimes |\downarrow\rangle}{\sqrt{2}}$$

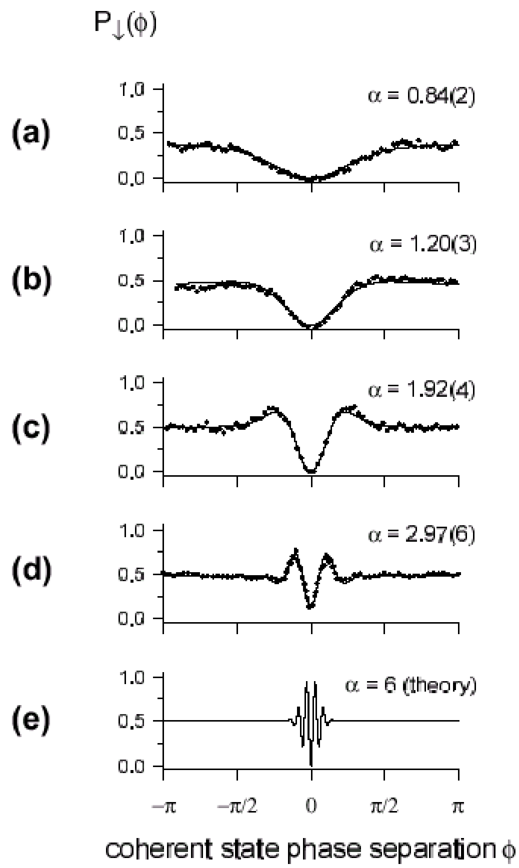
$$|\Psi_{(e)}\rangle = \frac{|-\alpha\rangle \otimes |\uparrow\rangle + |\alpha\rangle \otimes |\downarrow\rangle}{\sqrt{2}}$$



If the force F_2 displacement beam is such that it excites the $|0\rangle \otimes |\uparrow\rangle$ wave packet to the coherent state $|\alpha e^{i\phi}\rangle \otimes |\uparrow\rangle$, step (f) can then be applied to recombine the two spin components and then the signal can be recorded. The predicted signal as a function of ϕ for the state $|\Psi_{(f)}\rangle$ is:

$$P_{\downarrow}(\phi) = \frac{1}{2}\left[1 - ce^{-\alpha^2(1-\cos\phi)}\cos(\alpha^2\sin\phi)\right]$$

where c=1 in the absence of decoherence. Below are the results obtained from the experiment performed by C. Monroe, et. al. where the solid lines are fits to the predicted $P_{\downarrow}(\phi)$ (the decoherence for case (d) is large enough that c is significantly less than one and so the fit is adjusted for the loss of contrast).



To give a physical scale for this experiment, the coherent state $\alpha \approx 2.97$ corresponds to $\langle n \rangle \approx 9$ and has a maximum spatial separation $4\alpha x_0 \approx 83\ nm$, where the size of a single wave packet is $x_0 \approx 7.1\ nm$, and the atomic dimension is about $0.1\ nm$.

Here are a couple of references on Schrödinger Cat states:

P. Goetsch, R. Graham, and F. Haake, “Schrödinger cat states and single runs for the damped harmonic oscillator,” Phys. Rev. A 51, 136-142 (1995).

C. Monroe, D. M. Meekhof, B. E. King, and D. J. Wineland, “A ‘Schrödinger Cat’ Superposition State of an Atom,” Science 272, 1131-1136 (1996).