**Ph195a Final EXAM**

**Due Friday December 14 at 5:00 pm in the box outside 24 Bridge Annex**

- Closed book, and you may use only one page of prepared notes (both sides of a single 8.5”x11” sheet of paper).
- Use of computer software is not allowed on this exam.
- The time limit for this exam is three hours.
- ***Number in (parentheses) indicates points per problem.***

1. Let $|\Psi(t)\rangle$ represent an evolving quantum state in a four dimensional Hilbert space, $H = \text{span}\{|\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$.

(a) (10 points) Suppose $|\Psi(0)\rangle = \sum_j c_j|\phi_j\rangle$ and the Hamiltonian is $\mathbf{H}_0 = \varepsilon_0(\mathbf{1} + \mathbf{M})$, where

$$\mathbf{M} = |\phi_0\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_3| + |\phi_3\rangle\langle\phi_1|.$$

Derive a general expression for $|\Psi(t)\rangle$.

(b) (10 points) Suppose we now add an additional term $\varepsilon_r\left(\mathbf{R} + \mathbf{R}^\dagger\right)$ to the Hamiltonian, where

$$\mathbf{R} = |\phi_0\rangle\langle\phi_1| + |\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_3| + |\phi_3\rangle\langle\phi_0|.$$

Even though you don't have to, use degenerate perturbation theory to compute first-order (in $\varepsilon_r/\varepsilon_0$) corrections to the eigenstates and eigenvalues of $\mathbf{H}_0$.

(c) (10 points) Continuing from part (b), use the fact that $\left(\mathbf{R} + \mathbf{R}^\dagger\right)$ is a symmetry operator for $\mathbf{H}_0$ to prove that all higher-order (in $\varepsilon_r/\varepsilon_0$) corrections vanish.

(d) (10 points) Use perturbation theory to compute first-order (in $\lambda$) eigenvalue corrections for

$$\mathbf{H}_0 + \varepsilon_r\left(\mathbf{R} + \mathbf{R}^\dagger\right) \to \mathbf{H}_0 + \varepsilon_r\left(\mathbf{R} + \mathbf{R}^\dagger\right) + \lambda|\phi_0\rangle\langle\phi_0|.$$

You may assume that $\varepsilon_0 \neq \varepsilon_r \neq 0$.

2. The Bloch Equations with dissipation may be written

$$\frac{d}{dt}v_x = -\Delta v_y - v_x T_2^{-1},$$
$$\frac{d}{dt}v_y = \Delta v_x + \gamma b_1 v_z - v_y T_2^{-1},$$
$$\frac{d}{dt}v_z = -\gamma b_1 v_y - (v_z - v_z^0)T_1^{-1},$$

where $\vec{v} \equiv \langle\vec{\sigma}\rangle$. For this problem, set

$$\Delta = 0,$$
$$T_2 = \frac{1}{2}T_1,$$
$$\gamma b_1 T_1 \equiv \lambda \ll 1.$$

(a) (10 points) Find an initial density operator $\rho(t=0)$ such that the purity $\text{Tr}[\rho^2(t)]$ is constant in time. Note that the easiest way to do this is to find $\rho(t=0)$ such that $\rho(t) = \rho(0)$ at all times.

(b) (10 points) Find an initial density operator $\rho(0)$ such that the purity decreases with time, in the sense that $\text{Tr}[\rho^2(t)] < \text{Tr}[\rho^2(0)]$ for large $t$. You do not need to explicitly compute $\text{Tr}[\rho^2(t)]$, but you should provide a convincing argument that it decreases.

(c) (10 points) Find an initial density operator $\rho(0)$ such that the purity increases with time. Again you do not need to explicitly compute $\text{Tr}[\rho^2(t)]$, but you should provide a convincing argument that it increases.

(d) (10 points) Interpret the time-evolution of your answers to parts (b) and (c) for small $t$, in terms of dynamics on the Bloch sphere.