Ph 195a Final Exam Solutions:

Problem 1

(a) The first thing to do is find the eigenstates of $\mathbf{H}_0$. Since the identity operator commutes with any other operator, the eigenstates of $\mathbf{H}_0$ will simply be the eigenvectors $|\varphi_n^{i_n}\rangle$ of **M**, where $n = 0, 1$ and $i_n = 1, 2$. (For this problem, I am using notation that tries to match the notation used in the lecture on perturbation theory.) By inspection, these are:

$$|\varphi_0^1\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_3\rangle)$$
$$|\varphi_0^2\rangle = \frac{1}{\sqrt{2}}(|\phi_0\rangle - |\phi_2\rangle)$$
$$|\varphi_1^1\rangle = \frac{1}{\sqrt{2}}(|\phi_0\rangle + |\phi_2\rangle)$$
$$|\varphi_1^2\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_3\rangle)$$

These have the corresponding ($\mathbf{H}_0$) energy eigenvalues: $E_0^0 = 0$ and $E_1^0 = 2\varepsilon_0$. (Note that instead of $|\varphi_0^1\rangle$ and $|\varphi_0^2\rangle$ we could have chosen any orthogonal pair of linear combinations of these two states to be the eigenstates, and similarly for $|\varphi_1^1\rangle$ and $|\varphi_1^2\rangle$.)

Hence, the expression of evolution of a general initial state in this eigenbasis $|\Psi(0)\rangle = \sum_{n,i_n} a_n^{i_n}|\varphi_n^{i_n}\rangle$ is given by:

$$|\Psi(t)\rangle = \sum_{n,i_n} a_n^{i_n}\exp[-iE_n^0 t/\hbar]|\varphi_n^{i_n}\rangle = a_0^1|\varphi_0^1\rangle + a_0^2|\varphi_0^2\rangle + a_1^1\exp[-i2\varepsilon_0 t/\hbar]|\varphi_1^1\rangle + a_1^2\exp[-i2\varepsilon_0 t/\hbar]|\varphi_1^2\rangle$$

So the change of basis:

$$|\Psi(0)\rangle = \sum_j c_j|\phi_j\rangle = \frac{c_0}{\sqrt{2}}(|\varphi_0^2\rangle + |\varphi_1^1\rangle) + \frac{c_1}{\sqrt{2}}(|\varphi_0^1\rangle + |\varphi_1^2\rangle) + \frac{c_2}{\sqrt{2}}(-|\varphi_0^2\rangle + |\varphi_1^1\rangle) + \frac{c_3}{\sqrt{2}}(-|\varphi_0^1\rangle + |\varphi_1^2\rangle)$$
$$= \left(\frac{c_1-c_3}{\sqrt{2}}\right)|\varphi_0^1\rangle + \left(\frac{c_0-c_2}{\sqrt{2}}\right)|\varphi_0^2\rangle + \left(\frac{c_0+c_2}{\sqrt{2}}\right)|\varphi_1^1\rangle + \left(\frac{c_1+c_3}{\sqrt{2}}\right)|\varphi_1^2\rangle$$

gives the general expression for the time evolved state:

$$|\Psi(t)\rangle = \left(\frac{c_1-c_3}{\sqrt{2}}\right)|\varphi_0^1\rangle + \left(\frac{c_0-c_2}{\sqrt{2}}\right)|\varphi_0^2\rangle + \left(\frac{c_0+c_2}{\sqrt{2}}\right)\exp[-i2\varepsilon_0 t/\hbar]|\varphi_1^1\rangle + \left(\frac{c_1+c_3}{\sqrt{2}}\right)\exp[-i2\varepsilon_0 t/\hbar]|\varphi_1^2\rangle$$
$$= \left(\frac{c_1-c_3}{2}\right)(|\phi_1\rangle - |\phi_3\rangle) + \left(\frac{c_0-c_2}{2}\right)(|\phi_0\rangle - |\phi_2\rangle) + \left(\frac{c_0+c_2}{2}\right)\exp[-i2\varepsilon_0 t/\hbar](|\phi_0\rangle + |\phi_2\rangle)$$
$$+ \left(\frac{c_1+c_3}{2}\right)\exp[-i2\varepsilon_0 t/\hbar](|\phi_1\rangle + |\phi_3\rangle)$$
$$= \left(\frac{(c_0-c_2)+(c_0+c_2)\exp[-i2\varepsilon_0 t/\hbar]}{2}\right)|\phi_0\rangle + \left(\frac{(c_1-c_3)+(c_1+c_3)\exp[-i2\varepsilon_0 t/\hbar]}{2}\right)|\phi_1\rangle$$
$$+ \left(\frac{-(c_0-c_2)+(c_0+c_2)\exp[-i2\varepsilon_0 t/\hbar]}{2}\right)|\phi_2\rangle + \left(\frac{-(c_1-c_3)+(c_1+c_3)\exp[-i2\varepsilon_0 t/\hbar]}{2}\right)|\phi_3\rangle$$

Another way to do this is to compute the time evolution operator directly in the original basis, though the above method is more relavent to this problem, because we need the eigenstates and eigenvalues for the rest of the problem. First note that:

$$(\mathbf{H}_0)^2 = (\varepsilon_0)^2\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}^2 = 2(\varepsilon_0)^2\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = 2\varepsilon_0\mathbf{H}_0$$

and then we get:

$$\mathbf{T}_0(t,0) = \exp[-i\mathbf{H}_0 t/\hbar] = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-it/\hbar)^n(\mathbf{H}_0)^n}{n!} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-it/\hbar)^n(2\varepsilon_0)^n}{n!}\frac{1}{2}(\mathbf{1}+\mathbf{M})$$

$$= \frac{1}{2}(\mathbf{1}-\mathbf{M}) + \frac{1}{2}(\mathbf{1}+\mathbf{M}) + \sum_{n=1}^{\infty} \frac{(-it/\hbar)^n(2\varepsilon_0)^n}{n!}\frac{1}{2}(\mathbf{1}+\mathbf{M}) = \frac{1}{2}(\mathbf{1}-\mathbf{M}) + \sum_{n=0}^{\infty} \frac{(-it/\hbar)^n(2\varepsilon_0)^n}{n!}\frac{1}{2}(\mathbf{1}+\mathbf{M})$$

$$= \frac{1}{2}(\mathbf{1}-\mathbf{M}) + \exp[-i2\varepsilon_0 t/\hbar]\frac{1}{2}(\mathbf{1}+\mathbf{M})$$

So, applying this to the initial state directly gives the same result as above:

$$|\Psi(t)\rangle = \mathbf{T}_0(t,0)|\Psi(0)\rangle$$

$$= \begin{pmatrix} \frac{1+\exp[-i2\varepsilon_0 t/\hbar]}{2} & 0 & \frac{-1+\exp[-i2\varepsilon_0 t/\hbar]}{2} & 0 \\ 0 & \frac{1+\exp[-i2\varepsilon_0 t/\hbar]}{2} & 0 & \frac{-1+\exp[-i2\varepsilon_0 t/\hbar]}{2} \\ \frac{-1+\exp[-i2\varepsilon_0 t/\hbar]}{2} & 0 & \frac{1+\exp[-i2\varepsilon_0 t/\hbar]}{2} & 0 \\ 0 & \frac{-1+\exp[-i2\varepsilon_0 t/\hbar]}{2} & 0 & \frac{1+\exp[-i2\varepsilon_0 t/\hbar]}{2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(c_0-c_2)+(c_0+c_2)\exp[-i2\varepsilon_0 t/\hbar]}{2} \\ \frac{(c_1-c_3)+(c_1+c_3)\exp[-i2\varepsilon_0 t/\hbar]}{2} \\ \frac{-(c_0-c_2)+(c_0+c_2)\exp[-i2\varepsilon_0 t/\hbar]}{2} \\ \frac{-(c_1-c_3)+(c_1+c_3)\exp[-i2\varepsilon_0 t/\hbar]}{2} \end{pmatrix}$$

(b) Converting to the $\{|\varphi_n^{i_n}\rangle\}$ basis, we have:

$$\mathbf{H}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\varepsilon_0 & 0 \\ 0 & 0 & 0 & 2\varepsilon_0 \end{pmatrix}$$

$$\varepsilon_r(\mathbf{R}+\mathbf{R}^\dagger) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\varepsilon_r \\ 0 & 0 & 2\varepsilon_r & 0 \end{pmatrix}$$

$$\mathbf{H} = \mathbf{H}_0 + \varepsilon_r(\mathbf{R}+\mathbf{R}^\dagger) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\varepsilon_0 & 2\varepsilon_r \\ 0 & 0 & 2\varepsilon_r & 2\varepsilon_0 \end{pmatrix}$$

So we want to pick eigenstates of $\mathbf{H}_0$ that diagonalize the $\{|\varphi_0^i\rangle\}$ and $\{|\varphi_1^i\rangle\}$ subspaces (i.e. the degenerate subspaces of $\mathbf{H}_0$) of the perturbation matrix. The $\{|\varphi_0^i\rangle\}$ subspace is already (trivially) diagonal, so looking at the restriction to the $\{|\varphi_1^i\rangle\}$ subspace:

$$\varepsilon_r(\mathbf{R}+\mathbf{R}^\dagger)^{\{|\varphi_1^i\rangle\}} = \begin{pmatrix} 0 & 2\varepsilon_r \\ 2\varepsilon_r & 0 \end{pmatrix}$$

has eigenvectors: $\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$. So the desired eigenstates and their first order corrected eigenvalues are:

$|0^1\rangle = |\varphi_0^1\rangle \qquad E_0^1 = 0$

$|0^2\rangle = |\varphi_0^2\rangle \qquad E_0^2 = 0$

$|1^1\rangle = \frac{1}{\sqrt{2}}(|\varphi_1^1\rangle - |\varphi_1^2\rangle) \qquad E_1^1 = 2\varepsilon_0 - 2\varepsilon_r$

$|1^2\rangle = \frac{1}{\sqrt{2}}(|\varphi_1^1\rangle + |\varphi_1^2\rangle) \qquad E_1^2 = 2\varepsilon_0 + 2\varepsilon_r$

(The problem actually meant to ask for the zeroth-order eigenstates, but, as will be shown in part (c), the zeroth-order eigenstates are the same as the first-order (and higher) eigenstates.) Notice that this removes the degeneracy from only one of the two degenerate subspaces.

(c) Observe that **M** and $(\mathbf{R}+\mathbf{R}^\dagger)$ commute (calculation done in the original basis):

$$\mathbf{M}(\mathbf{R}+\mathbf{R}^\dagger) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$(\mathbf{R}+\mathbf{R}^\dagger)\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Hence, we can find simultaneous eigenvectors for these two matrices. In particular, the zeroth-order eigenstates

$|0^1\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_3\rangle)$

$|0^2\rangle = \frac{1}{\sqrt{2}}(|\phi_0\rangle - |\phi_2\rangle)$

$|1^1\rangle = \frac{1}{2}(|\phi_0\rangle - |\phi_1\rangle + |\phi_2\rangle - |\phi_3\rangle)$

$|1^2\rangle = \frac{1}{2}(|\phi_0\rangle + |\phi_1\rangle + |\phi_2\rangle + |\phi_3\rangle)$

are just such simultaneous eigenstates of $\mathbf{H}_0$ and $\varepsilon_r(\mathbf{R}+\mathbf{R}^\dagger)$, and so they are also eigenstates of **H**. Thus, this choice of eigenstates $\{|0^i\rangle, |1^i\rangle\}$ as a basis not only diagonalized the $\{|\varphi_0^i\rangle\}$ and $\{|\varphi_1^i\rangle\}$ subspaces of the perturbation, but also diagonalized the entire (perturbed) Hamiltonian matrix. This means that all $\langle m^k|(\mathbf{R}+\mathbf{R}^\dagger)|n^i\rangle$ terms vanish unless $m = n$ and $k = i$ and that the first order corrected eigenvalues found above coincide with the exact eigenvalues of $\mathbf{H}_0 + \varepsilon_r(\mathbf{R}+\mathbf{R}^\dagger)$. It follows that higher order corrections vanish (i.e. higher than zeroth-order for eigenstates and first-order for eigenvalues).

(d) Now define the eigenbasis $\{|\xi_0^1\rangle, |\xi_0^2\rangle, |\xi_1\rangle, |\xi_2\rangle\}$ of $\mathbf{H}_0 + \varepsilon_r(\mathbf{R}+\mathbf{R}^\dagger)$, and their eigenvalues:

$|\xi_0^1\rangle = |0^1\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_3\rangle) \qquad E_0^1 = 0$

$|\xi_0^2\rangle = |0^2\rangle = \frac{1}{\sqrt{2}}(|\phi_0\rangle - |\phi_2\rangle)$ $\qquad E_0^2 = 0$

$|\xi_1\rangle = |1^1\rangle = \frac{1}{2}(|\phi_0\rangle - |\phi_1\rangle + |\phi_2\rangle - |\phi_3\rangle)$ $\qquad E_1 = 2\varepsilon_0 - 2\varepsilon_r$

$|\xi_2\rangle = |1^2\rangle = \frac{1}{2}(|\phi_0\rangle + |\phi_1\rangle + |\phi_2\rangle + |\phi_3\rangle)$ $\qquad E_2 = 2\varepsilon_0 + 2\varepsilon_r$

Then, for a perturbation $\mathbf{W} = \lambda|\phi_0\rangle\langle\phi_0|$, we see that the restriction of **W** to the $\{|\xi_0^i\rangle\}$ subspace (i.e. the degenerate subspace of $\mathbf{H}_0 + \varepsilon_r(\mathbf{R} + \mathbf{R}^\dagger)$) is:

$$\mathbf{W}^{\{|\xi_0^i\rangle\}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\lambda \end{pmatrix}$$

which is already diagonal, so we can apply regular perturbation theory to get the corrections: $\Delta E_n^i = \langle\xi_n^i|\mathbf{W}|\xi_n^i\rangle = \lambda|\langle\xi_n^i|\phi_0\rangle|^2$

So, the first order corrected energy eigenvalues are:

$\langle\xi_0^1|\mathbf{W}|\xi_0^1\rangle = 0$ $\qquad E_0^1 = 0$

$\langle\xi_0^2|\mathbf{W}|\xi_0^2\rangle = \frac{1}{2}\lambda$ $\qquad E_0^2 = \frac{1}{2}\lambda$

$\langle\xi_1|\mathbf{W}|\xi_1\rangle = \frac{1}{4}\lambda$ $\qquad E_1 = 2\varepsilon_0 - 2\varepsilon_r + \frac{1}{4}\lambda$

$\langle\xi_2|\mathbf{W}|\xi_2\rangle = \frac{1}{4}\lambda$ $\qquad E_2 = 2\varepsilon_0 + 2\varepsilon_r + \frac{1}{4}\lambda$

Notice that this removes the remaining degeneracy.

Problem 2

I will first solve the differential equations for the evolution of the Bloch vector for this system in terms of general inital conditions, and then the desired answers can be found by inspection when we recall the two relations between the density operator and the Bloch vector: $\boldsymbol{\rho} = \frac{1}{2}\left(\mathbf{1} + \vec{v}\cdot\vec{\sigma}\right)$ and $\mathbf{Tr}[\boldsymbol{\rho}^2] = \frac{1}{2}\left(1 + |\vec{v}|^2\right)$. So the system of differential equations we want to solve is:

$$\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} -2T_1^{-1} & 0 & 0 \\ 0 & -2T_1^{-1} & \gamma b_1 \\ 0 & -\gamma b_1 & -T_1^{-1} \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ T_1^{-1}v_z^0 \end{pmatrix}$$

Solving for the homogeneous solution:

$$\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} + \begin{pmatrix} 2T_1^{-1} & 0 & 0 \\ 0 & 2T_1^{-1} & -\gamma b_1 \\ 0 & \gamma b_1 & T_1^{-1} \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0 = \det\begin{pmatrix} -\eta + 2T_1^{-1} & 0 & 0 \\ 0 & -\eta + 2T_1^{-1} & -\gamma b_1 \\ 0 & \gamma b_1 & -\eta + T_1^{-1} \end{pmatrix} = (-\eta + 2T_1^{-1})\left((-\eta + 2T_1^{-1})(-\eta + T_1^{-1}) + (\gamma b_1)^2\right)$$

$$0 = (-\eta + 2T_1^{-1})\left(\eta^2 - 3T_1^{-1}\eta + 2T_1^{-2} + (\gamma b_1)^2\right)$$

$$\eta_1 = 2T_1^{-1} \qquad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\eta_2 = \left(\frac{3}{2} + \frac{1}{2}\sqrt{1-(2\gamma b_1 T_1)^2}\right)T_1^{-1} \qquad \vec{e}_2 = \begin{pmatrix} 0 \\ \frac{\gamma b_1 T_1}{\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-(2\gamma b_1 T_1)^2}\right)^{\frac{1}{2}}} \\ \frac{\frac{1}{2}-\frac{1}{2}\sqrt{1-(2\gamma b_1 T_1)^2}}{\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-(2\gamma b_1 T_1)^2}\right)^{\frac{1}{2}}} \end{pmatrix}$$

$$\eta_3 = \left(\frac{3}{2} - \frac{1}{2}\sqrt{1-(2\gamma b_1 T_1)^2}\right)T_1^{-1} \qquad \vec{e}_3 = \begin{pmatrix} 0 \\ \frac{\gamma b_1 T_1}{\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-(2\gamma b_1 T_1)^2}\right)^{\frac{1}{2}}} \\ \frac{\frac{1}{2}+\frac{1}{2}\sqrt{1-(2\gamma b_1 T_1)^2}}{\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-(2\gamma b_1 T_1)^2}\right)^{\frac{1}{2}}} \end{pmatrix}$$

$$\vec{v}_h(t) = c_1\vec{e}_1\exp(-\eta_1 t) + c_2\vec{e}_2\exp(-\eta_2 t) + c_3\vec{e}_3\exp(-\eta_3 t)$$

Letting $\lambda \equiv \gamma b_1 T_1 \ll 1$, and noticing that $\eta_i > 0$ for all i, we can write this in a more suggestive form by calling $\tau_j = \eta_j^{-1}$, getting:

$$\tau_1 = \frac{1}{2}T_1 \qquad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\tau_2 = \left(\frac{3}{2} + \frac{1}{2}\sqrt{1-4\lambda^2}\right)^{-1}T_1 \qquad \vec{e}_2 = \begin{pmatrix} 0 \\ \frac{\lambda}{\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-4\lambda^2}\right)^{\frac{1}{2}}} \\ \frac{\frac{1}{2}-\frac{1}{2}\sqrt{1-4\lambda^2}}{\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-4\lambda^2}\right)^{\frac{1}{2}}} \end{pmatrix}$$

$$\tau_3 = \left(\frac{3}{2} - \frac{1}{2}\sqrt{1-4\lambda^2}\right)^{-1}T_1 \qquad \vec{e}_3 = \begin{pmatrix} 0 \\ \frac{\lambda}{\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\lambda^2}\right)^{\frac{1}{2}}} \\ \frac{\frac{1}{2}+\frac{1}{2}\sqrt{1-4\lambda^2}}{\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\lambda^2}\right)^{\frac{1}{2}}} \end{pmatrix}$$

$$\vec{v}_h(t) = c_1\vec{e}_1\exp(-t/\tau_1) + c_2\vec{e}_2\exp(-t/\tau_2) + c_3\vec{e}_3\exp(-t/\tau_3)$$

The inhomogeneous (particular) solution is exactly the steady state solution, since a constant (time independent) inhomogeneous solution is appropriate for this system. For the stationary state solution, let $\frac{d}{dt}\vec{v} = 0$, then:

$$\begin{pmatrix} 2T_1^{-1} & 0 & 0 \\ 0 & 2T_1^{-1} & -\gamma b_1 \\ 0 & \gamma b_1 & T_1^{-1} \end{pmatrix} \begin{pmatrix} v_x^{ss} \\ v_y^{ss} \\ v_z^{ss} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ T_1^{-1} v_z^0 \end{pmatrix}$$

$$\begin{pmatrix} v_x^{ss} \\ v_y^{ss} \\ v_z^{ss} \end{pmatrix} = \begin{pmatrix} 2T_1^{-1} & 0 & 0 \\ 0 & 2T_1^{-1} & -\gamma b_1 \\ 0 & \gamma b_1 & T_1^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ T_1^{-1} v_z^0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2T_1^{-1}} & 0 & 0 \\ 0 & \frac{T_1^{-1}}{2T_1^{-2}+(\gamma b_1)^2} & \frac{\gamma b_1}{2T_1^{-2}+(\gamma b_1)^2} \\ 0 & \frac{-\gamma b_1}{2T_1^{-2}+(\gamma b_1)^2} & \frac{2T_1^{-1}}{2T_1^{-2}+(\gamma b_1)^2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ T_1^{-1} v_z^0 \end{pmatrix}$$

$$\begin{pmatrix} v_x^{ss} \\ v_y^{ss} \\ v_z^{ss} \end{pmatrix} = \begin{pmatrix} 0 \\ \left(\frac{\gamma b_1 T_1}{2+(\gamma b_1 T_1)^2}\right) v_z^0 \\ \left(\frac{2}{2+(\gamma b_1 T_1)^2}\right) v_z^0 \end{pmatrix} = \begin{pmatrix} 0 \\ \left(\frac{\lambda}{2+\lambda^2}\right) v_z^0 \\ \left(\frac{2}{2+\lambda^2}\right) v_z^0 \end{pmatrix}$$

So we now have the general form for the time dependent Bloch vector:
$\vec{v}(t) = \vec{v}_h(t) + \vec{v}_p = c_1 \vec{e}_1 \exp(-t/\tau_1) + c_2 \vec{e}_2 \exp(-t/\tau_2) + c_3 \vec{e}_3 \exp(-t/\tau_3) + \vec{v}^{ss}$

for which the constants c_1, c_2, c_3 are chosen to satisfy the initial conditions (i.e. $\vec{v}(0) = c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3 + \vec{v}^{ss}$). Solving for these constants is done by inverting the matrix

composed of the i^{th} eigenvector as the i^{th} column, and then multiplying this inverse by $\vec{v}(0) - \vec{v}^{ss}$. This ends up being a very messy calculation for all but a few well picked inital conditions, so I will not go further with it. (It is not necessary to know exactly what these constants are to understand the general behavior of the time evolution.) We can check that the results make sense, by taking the limit $\lambda \to 0$ and observing that:

$$\tau_1 \to \tfrac{1}{2} T_1 = T_2 \qquad \vec{e}_1 \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\tau_2 \to \tfrac{1}{2} T_1 = T_2 \qquad \vec{e}_2 \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\tau_3 \to T_1 \qquad \vec{e}_3 \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{v}^{ss} \to \begin{pmatrix} 0 \\ 0 \\ v_z^0 \end{pmatrix}$$

are exactly the results we would get if $\lambda = 0$ (i.e. if there was no coupling).

It should be clear at this point that (since $\tau_i > 0$ for all i) the time evolution of the Bloch vector components follow linear combinations of exponential decays (which can be increasing or decreasing, depending on the initial conditions) from any initial state to the steady state solution. This gives the important result: **All initial states evolve to the steady state solution $\vec{v}^{ss}$ as $t \to \infty$.** We can now pick out the answers to this question:

(a) The purity will be constant whenever the magnitude of the Bloch vector will be constant in time. Hence, the obvious solution is to pick the steady state solution of the Bloch equations, so that $\boldsymbol{\rho}(t) = \boldsymbol{\rho}(0)$:

$$\boldsymbol{\rho}(0) = \boldsymbol{\rho}^{ss} = \frac{1}{2}\left(\mathbf{1} + \vec{v}^{ss} \cdot \vec{\sigma}\right) = \begin{pmatrix} \frac{1}{2} + \left(\frac{v_z^0}{2+\lambda^2}\right) & -\frac{i}{2}\left(\frac{\lambda v_z^0}{2+\lambda^2}\right) \\ \frac{i}{2}\left(\frac{\lambda v_z^0}{2+\lambda^2}\right) & \frac{1}{2} - \left(\frac{v_z^0}{2+\lambda^2}\right) \end{pmatrix}$$

Note that

$$\left|\vec{v}^{ss}\right| = \left(\frac{4+\lambda^2}{(2+\lambda^2)^2}\right)^{\frac{1}{2}} v_z^0$$

$$\mathbf{Tr}\left[(\boldsymbol{\rho}^{ss})^2\right] = \frac{1}{2}\left(1 + \left|\vec{v}^{ss}\right|^2\right) = \frac{1}{2}\left(1 + \frac{4+\lambda^2}{(2+\lambda^2)^2}(v_z^0)^2\right) < 1$$

since we are assumming that $|v_z^0| \leq 1$ in order for it to physically make sense.

(b) Since the Bloch vector and density operator evolve toward their steady state values, the purity will be arbitrarily close to the steady state purity for large enough t. Hence, the purity will decrease with time in the sense described in the problem for any initial state with $\left|\vec{v}(0)\right|$ larger than $\left|\vec{v}^{ss}\right|$. (Note that the evolution can have decreasing purity in the sense described in the problem without having to be uniformly or monotonically decreasing.) This condition is satified by many possible cases, the most obvious being the pure states (i.e. any Bloch vector with magnitude 1 has a purity of 1). For example, $\vec{v}(0) = (0, 0, 1)$

$$\boldsymbol{\rho}(0) = \frac{1}{2}\left(\mathbf{1} + \vec{v}(0) \cdot \vec{\sigma}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

If you did not want to go through the trouble of solving the entire system of differential equations (i.e. the Bloch equations) in order to know that all solutions evolved to the steady state, you could still get a solution for this part by noting that the x component is decoupled from the y and z components of the equations, and the x component equation is easy to solve by itself (i.e. regular exponential decay). Thus, picking the y and z components of the initial Bloch vector to be the y and z components of the steady state solution will leave them unchanged throughout time, and any non-zero x component will simply decay away. The only thing to be careful about is to make sure that the initial x component is small enough that the Bloch vector is not unphysical, i.e. make sure that: $0 < (v_x(0))^2 \leq 1 - \left|\vec{v}^{ss}\right|^2$. Then you have:

$$\vec{v}(t) = \begin{pmatrix} v_x(0)\exp(-t/\tau_1) \\ \left(\frac{\lambda}{2+\lambda^2}\right) v_z^0 \\ \left(\frac{2}{2+\lambda^2}\right) v_z^0 \end{pmatrix}$$

$$\boldsymbol{\rho}(t) = \frac{1}{2}\left(\mathbf{1} + \vec{v}(t)\cdot\vec{\sigma}\right) = \begin{pmatrix} \frac{1}{2} + \left(\frac{v_z^0}{2+\lambda^2}\right) & -\frac{i}{2}\left(\frac{\lambda v_z^0}{2+\lambda^2}\right) + \frac{v_x(0)}{2}\exp(-t/\tau_1) \\ \frac{i}{2}\left(\frac{\lambda v_z^0}{2+\lambda^2}\right) + \frac{v_x(0)}{2}\exp(-t/\tau_1) & \frac{1}{2} - \left(\frac{v_z^0}{2+\lambda^2}\right) \end{pmatrix}$$

So you would pick:

$$\boldsymbol{\rho}(0) = \begin{pmatrix} \frac{1}{2} + \left(\frac{v_z^0}{2+\lambda^2}\right) & -\frac{i}{2}\left(\frac{\lambda v_z^0}{2+\lambda^2}\right) + \frac{v_x(0)}{2} \\ \frac{i}{2}\left(\frac{\lambda v_z^0}{2+\lambda^2}\right) + \frac{v_x(0)}{2} & \frac{1}{2} - \left(\frac{v_z^0}{2+\lambda^2}\right) \end{pmatrix}$$

(c) By the same reasoning in part (b), the purity will increase with time if $\left|\vec{v}(0)\right|$ is smaller than $\left|\vec{v}^{ss}\right|$. As long as $v_z^0 \neq 0$ (i.e. thermal equilibrium in the z-component is not isotropic) then $\left|\vec{v}^{ss}\right| > 0$, and this initial condition can also be satisfied by many possible cases. The most obvious choice is $\vec{v}(0) = (0,0,0)$, which corresponds to the completely isotropic state (i.e. a purity of $\frac{1}{2}$):

$$\boldsymbol{\rho}(0) = \frac{1}{2}\left(\mathbf{1} + \vec{v}(0)\cdot\vec{\sigma}\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Again, without solving the differential equations, $\vec{v}(0) = (0,0,0)$ should be the first guess, since it is clear that this Bloch vector is the only Bloch vector with the minimum possible purity. Then, one only needs to provide a convincing arguement for why this initial state will not be in this state at any subsequent time in the evolution of the system.

(d) The interpretation of the evolution of the answers from part (b) and (c) depends on the examples provided, but the general interpretation for small t evolution on the Bloch sphere can be read off the Bloch equations. The Bloch vector (v_x, v_y, v_z) is primarily evolving towards $(0,0,v_z^0)$ with decay times (T_2, T_2, T_1) respectively, but there is also a coupling between the y and z components which tends to rotate the the Bloch vector towards the equatorial (xy) plane of the Bloch sphere, and so gives evolution to a steady state that is slightly rotated (and contracted a little) from $(0,0,v_z^0)$ in the y-direction.