Ph 195b Final Solutions:

Problem 1

Solve for the general solutions in region 2 ($x < 0$) and region 2 ($0 < x < L$) and match the boundary conditions for a stationary (i.e. constant energy) state. Letting $k = \sqrt{\frac{2mE}{\hbar^2}}$, the solution for the free Schrödinger equation is of the form: $Ae^{ikx} + Be^{-ikx}$, so we have:

$\Psi_1(x) = Ae^{ikx} + Be^{-ikx}$

$\Psi_2(x) = Ce^{ikx} + De^{-ikx}$

Boundary conditions:

$\Psi_2(L) = 0 \Rightarrow Ce^{ikL} + De^{-ikL} = 0 \Rightarrow D = -Ce^{i2kL}$

$\Psi_1(0) = \Psi_2(0) \Rightarrow A + B = C + D = (1 - e^{i2kL})C$

$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \left\{ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi(x) + V(x)\Psi(x) = E\Psi(x) \right\} \Rightarrow -\frac{\hbar^2}{2m} \left(\frac{d\Psi(0^+)}{dx} - \frac{d\Psi(0^-)}{dx} \right) + g\Psi(0) = 0$

$\Rightarrow -\frac{\hbar^2}{2m} ik(C - D - A + B) = -g(C + D) \Rightarrow A - B = \left(1 + e^{i2kL} + i\frac{2mg}{\hbar^2 k}(1 - e^{i2kL})\right)C$

$A = \left(1 + i\frac{mg}{\hbar^2 k}(1 - e^{i2kL})\right)C$

$B = \left(-e^{i2kL} - i\frac{mg}{\hbar^2 k}(1 - e^{i2kL})\right)C$

$D = (-e^{i2kL})C$

I will multiply these solutions by an overall factor of $\frac{1}{2i}e^{-ikL}$ (i.e. redefine the undetermined constant C) to give a cleaner looking solution:

$$\Psi(x) = \begin{cases} C\left(\sin[k(x-L)] + \frac{2mg}{\hbar^2 k}\sin[kL]\sin[kx]\right) & \text{for } x < 0 \\ C\sin[k(x-L)] & \text{for } 0 < x < L \end{cases}$$

Alternatively, I can write the solution in a form more suggestive of a scattering process:

$$\Psi(x) = \begin{cases} A(e^{ikx} + \beta e^{-ikx}) & \text{for } x < 0 \\ A(\gamma e^{ikx} - \gamma e^{i2kL}e^{-ikx}) = A(2i\gamma e^{ikL})\sin[k(x-L)] & \text{for } 0 < x < L \end{cases}$$

where $\beta = \frac{\left(-e^{i2kL} - i\frac{mg}{\hbar^2 k}(1 - e^{i2kL})\right)}{\left(1 + i\frac{mg}{\hbar^2 k}(1 - e^{i2kL})\right)}$ and $\gamma = \frac{1}{\left(1 + i\frac{mg}{\hbar^2 k}(1 - e^{i2kL})\right)}$. (Notice $|\beta|^2 = 1$ as expected.)

Problem 2

(a) Writing out the potential as:

$V(x,y) = g[\delta(x-1)\delta(y) + \delta(x)\delta(y-1) + \delta(x+1)\delta(y) + \delta(x)\delta(y+1)]$

$V(\rho,\varphi) = g\frac{\delta(\rho-1)}{\rho}\left[\delta(\varphi) + \delta\left(\varphi - \frac{\pi}{2}\right) + \delta(\varphi - \pi) + \delta\left(\varphi - \frac{3\pi}{2}\right)\right]$

we can see that the full symmetry group of this Hamiltonian is the dihedral group D_4. For those of you not familiar with the dihedral group D_n, it is the (unique up to isomorphism) group of order $2n$ generated by two elements p and r which obey the relations: $r \cdot p = p \cdot r^{-1}$, $p^2 = r^n = 1$, and $r^k \neq 1$ for $0 < k < n$. Relating this to the physical description of the group acting on a regular n-gon, r corresponds to a rotation by $\frac{2\pi}{n}$ and p corresponds to the reflection about one of the symmetry axes. In the case of $n = 4$, these are the actions on a square of rotation by $\frac{\pi}{2}$ and reflection about one of the symmetry axes (x-axis, y-axis, (xy)-axis, or $(-xy)$-axis).

Hence, there are essentially two important symmetries for this system (and all the other ones can be generated from these two). The symmetry actions that I will choose to look at are counter-clockwise rotation and reflection about the x-axis. Their corresponding operators and actions on the coordinates (x,y) or (ρ,φ) can be given by:

$R_\varphi\left(\frac{\pi}{2}\right) = \exp\left[-\frac{i}{\hbar}\mathbf{L}\frac{\pi}{2}\right] = \exp\left[-\frac{\pi}{2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] = \exp\left[-\frac{\pi}{2}\frac{\partial}{\partial\varphi}\right] : x \mapsto -y, y \mapsto x$ or

$\rho \mapsto \rho, \varphi \mapsto \varphi + \frac{\pi}{2}$

$\mathbf{\Pi}_y : x \mapsto x, y \mapsto -y$

Notice that $\mathbf{\Pi}_y$ is not the same thing as the parity operator $\mathbf{\Pi} : x \mapsto -x, y \mapsto -y$ (which actually is the same thing as $R_\varphi(\pi) = \left(R_\varphi\left(\frac{\pi}{2}\right)\right)^2$). Additionally, $\mathbf{\Pi}_y$ does not have a corresponding differential operator and there is no really clean way to write the action of $\mathbf{\Pi}_y$ on polar coordinates. Also, note that $R_\varphi\left(\frac{\pi}{2}\right)$ and $\mathbf{\Pi}_y$ do not commute (the group generators of the dihedral group do not commute by definition), so they do not have simultaneous eigenstates. Obviously, we are more interested in the rotation symmetry than the reflection symmetry since it provides more symmetry to simplify the problem with, but it is good to keep both in mind.

(b) I will show commutation of $R_\varphi\left(\frac{\pi}{2}\right)$ with the Hamiltonian by using the group action of the operators rather than the differential action. (I will also show that $\mathbf{\Pi}_y$ commutes with the Hamiltonian as well.)

By construction, the operators $R_\varphi\left(\frac{\pi}{2}\right)$ and $\mathbf{\Pi}_y$ commute with the potential, but I will show it anyways:

$$V\left(\rho,\varphi+\frac{\pi}{2}\right) = g\frac{\delta(\rho-1)}{\rho}\left[\delta\left(\varphi+\frac{\pi}{2}\right)+\delta(\varphi)+\delta\left(\varphi-\frac{\pi}{2}\right)+\delta(\varphi-\pi)\right]$$
$$= g\frac{\delta(\rho-1)}{\rho}\left[\delta\left(\varphi-\frac{3\pi}{2}\right)+\delta(\varphi)+\delta\left(\varphi-\frac{\pi}{2}\right)+\delta(\varphi-\pi)\right] = V(\rho,\varphi)$$
$$V(x,-y) = g[\delta(x-1)\delta(-y)+\delta(x)\delta(-y-1)+\delta(x+1)\delta(-y)+\delta(x)\delta(-y+1)]$$
$$= g[\delta(x-1)\delta(y)+\delta(x)\delta(y+1)+\delta(x+1)\delta(y)+\delta(x)\delta(y-1)] = V(x,y)$$
$$R_\varphi\left(\frac{\pi}{2}\right)V(\rho,\varphi) = V\left(\rho,\varphi+\frac{\pi}{2}\right)R_\varphi\left(\frac{\pi}{2}\right) = V(\rho,\varphi)R_\varphi\left(\frac{\pi}{2}\right)$$
$$\mathbf{\Pi}_yV(x,y) = V(x,-y)\mathbf{\Pi}_y = V(x,y)\mathbf{\Pi}_y$$

So it remains to show that these operators commute with ∇^2:

$$\frac{\partial}{\partial\left(\varphi+\frac{\pi}{2}\right)} = \frac{\partial}{\partial\varphi}$$
$$\frac{\partial}{\partial(-y)} = -\frac{\partial}{\partial y}$$
$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$$
$$R_\varphi\left(\frac{\pi}{2}\right)\nabla^2 = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\left(\varphi+\frac{\pi}{2}\right)^2}\right)R_\varphi\left(\frac{\pi}{2}\right)$$
$$= \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\left(\varphi+\frac{\pi}{2}\right)^2}\right)R_\varphi\left(\frac{\pi}{2}\right) = \nabla^2R_\varphi\left(\frac{\pi}{2}\right)$$
$$\mathbf{\Pi}_y\nabla^2 = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial(-y)^2}\right)\mathbf{\Pi}_y = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\mathbf{\Pi}_y = \nabla^2\mathbf{\Pi}_y$$

(c) The class of solutions that takes only a little work to find are the solutions of the free Schrödinger equation that vanish at the four points $(1,0),(0,1),(-1,0),(0,-1)$. The reason why these are also solutions of the full Hamiltonian is because the solutions of the full

Hamiltonian will be free particle solutions in the regions not including these points, will be continuous everywhere, and have first derivative continuity everywhere except at these four points where there is a derivative discontinuity proportional to the value of the wavefunction at these points. Hence, if the wave function vanishes at these points, the first derivative discontinuities will be zero, and so the full Hamiltonian solution will also have derivative continuity everywhere, i.e. it will be exactly the same as a free particle solution (that vanishes at these four points). Here are the solutions of this type:

In polar coordinates, the solution to the free (2-D) Schrödinger equation is of the form: $\sum_{n=-\infty}^{\infty}(A_nJ_n(k\rho)+B_nN_n(k\rho))e^{in\varphi}$. Since $N_n(k\rho)$ all diverge at $\rho=0$, we must set $B_n=0$. Using $J_{-n}=(-1)^nJ_n$, we can write this as:

$$\Psi_k(\rho,\varphi)=A_0J_0(k\rho)+\sum_{n=1}^{\infty}J_n(k\rho)(A_ne^{in\varphi}+C_ne^{-in\varphi})$$

For this to vanish at the desired points, we must have:

$A_0=0$ unless $\{k$ is a root of $J_0\}$

$A_n=C_n=0$ unless $\{k$ is a root of $J_n\}$ or $\{C_n=-A_n$ and n is even$\}$

Since $\left(R_\varphi\left(\frac{\pi}{2}\right)\right)^4=\mathbf{1}$, we can see that an eigenstate of $R_\varphi\left(\frac{\pi}{2}\right)$ must have eigenvalue $\lambda=1,i,-1,$ or $-i$. Hence, the $k=\sqrt{\frac{2mE}{\hbar^2}},\lambda_{R_\varphi\left(\frac{\pi}{2}\right)}=1$ eigenstates are given by linear combinations of:

$J_n(k\rho)\sin[n\varphi]$ when $n=4m$ for $m\in\mathbf{Z}^+$

$J_0(k\rho)$ when k is a root of J_0

$J_n(k\rho)e^{in\varphi}$ when k is a root of J_n and $n=4m$ for $m\in\mathbf{Z}^+$

$J_n(k\rho)e^{-in\varphi}$ when k is a root of J_n and $n=4m$ for $m\in\mathbf{Z}^+$

the $k=\sqrt{\frac{2mE}{\hbar^2}},\lambda_{R_\varphi\left(\frac{\pi}{2}\right)}=i$ eigenstates are given by linear combinations of:

$J_n(k\rho)e^{in\varphi}$ when k is a root of J_n and $n=4m+1$ for $m\in\mathbf{Z}^+$

$J_n(k\rho)e^{-in\varphi}$ when k is a root of J_n and $n=4m+3$ for $m\in\mathbf{Z}^+$

the $k=\sqrt{\frac{2mE}{\hbar^2}},\lambda_{R_\varphi\left(\frac{\pi}{2}\right)}=-1$ eigenstates are given by linear combinations of:

$J_n(k\rho)\sin[n\varphi]$ when $n=4m+2$ for $m\in\mathbf{Z}^+$

$J_n(k\rho)e^{in\varphi}$ when k is a root of J_n and $n=4m+2$ for $m\in\mathbf{Z}^+$

$J_n(k\rho)e^{-in\varphi}$ when k is a root of J_n and $n=4m+2$ for $m\in\mathbf{Z}^+$

and, the $k=\sqrt{\frac{2mE}{\hbar^2}},\lambda_{R_\varphi\left(\frac{\pi}{2}\right)}=-i$ eigenstates are given by linear combinations of:

$J_n(k\rho)e^{in\varphi}$ when k is a root of J_n and $n=4m+3$ for $m\in\mathbf{Z}^+$

$J_n(k\rho)e^{-in\varphi}$ when k is a root of J_n and $n=4m+1$ for $m\in\mathbf{Z}^+$

In cartesian coordinates, the solution to the free Schrödinger equation is of the form:

$$\Psi_k(x,y)=(A\cos[k_xx]+B\sin[k_xx])(C\cos[k_yy]+D\sin[k_yy])$$

where $k_x^2+k_y^2=k^2$ (and k_x,k_y need not be real-valued). For this to vanish at the desired points, we must have:

$A=0$ unless $\{k_x$ and k_y are roots of $\cos$ and $B=D=0\}$ or $\{k_y$ is root of $\sin$ and $C=0\}$

$B=0$ unless $\{k_x$ is root of $\sin$ and $A=0\}$ or $\{k_y$ is root of $\sin$ and $C=0\}$ or $\{A=C=0\}$

$C=0$ unless $\{k_x$ and k_y are roots of $\cos$ and $B=D=0\}$ or $\{k_x$ is root of $\sin$ and $A=0\}$

$D = 0$ unless $\{k_y$ is root of $\sin$ and $C = 0\}$ or $\{k_x$ is root of $\sin$ and $A = 0\}$ or $\{A = C = 0\}$
The eigenvalues of $\Pi_y$are $\lambda_{\Pi_y} = 1, -1$ since Π_y^2= **1**. So the $k = (k_x^2 + k_y^2)^{\frac{1}{2}} = \sqrt{\frac{2mE}{\hbar^2}}, \lambda_{\Pi_y} = 1$ eigenstates in this class of solutions are given by linear combinations of:
$\cos[k_x x]\cos[k_y y]$ when k_x and k_y are roots of cosine (i.e. $k_x = \left(n + \frac{1}{2}\right)\pi, k_y = \left(m + \frac{1}{2}\right)\pi$ for $n, m \in$ **Z**)
$\sin[k_x x]\cos[k_y y]$ when k_x is a root of sine (i.e. $k_x = n\pi$ for $n \in$ **Z**)
and the $k = (k_x^2 + k_y^2)^{\frac{1}{2}} = \sqrt{\frac{2mE}{\hbar^2}}, \lambda_{\Pi_y} = -1$ eigenstates in this class of solutions are given by linear combinations of:
$\sin[k_x x]\sin[k_y y]$
$\cos[k_x x]\sin[k_y y]$ when k_y is a root of sine (i.e. $k_y = m\pi$ for $m \in$ **Z**)

Problem 3

(a) $J_{tot} = J_A + J_B, \ldots, |J_A - J_B| = 2, 1, 0$

(b) $|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle_A + |1,-1\rangle_A) \otimes |1,1\rangle_B = \frac{1}{\sqrt{2}}(|1,1\rangle_A \otimes |1,1\rangle_B + |1,-1\rangle_A \otimes |1,1\rangle_B)$
$|1,1\rangle_A \otimes |1,1\rangle_B = |2,2\rangle_{tot}$
$|1,-1\rangle_A \otimes |1,1\rangle_B = \frac{1}{\sqrt{6}}|2,0\rangle_{tot} - \frac{1}{\sqrt{2}}|1,0\rangle_{tot} + \frac{1}{\sqrt{3}}|0,0\rangle_{tot}$
$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}|2,2\rangle_{tot} + \frac{1}{\sqrt{12}}|2,0\rangle_{tot} - \frac{1}{\sqrt{4}}|1,0\rangle_{tot} + \frac{1}{\sqrt{6}}|0,0\rangle_{tot}$
I looked these CG-coefficients in a table that was more convenient than the one provided on the exam (i.e. Griffiths' *Introduction to Quantum Mechanics*, pg 168). You could only get the coefficients of $|1,0\rangle_{tot}$ and $|0,0\rangle_{tot}$ from the table provided (using $C^{c,\alpha+\beta}_{a,\alpha;b,\beta} = (-1)^{a+b-c} C^{c,\alpha+\beta}_{b,\beta;a,\alpha}$). If the CG table had one more row (i.e. if it included $C^{1,1}_{2,0;1,1} = \frac{1}{\sqrt{10}}$), you would have been able to get the remaining coefficient using $C^{c,\alpha+\beta}_{a,\alpha;b,\beta} = (-1)^{a-b+\alpha+\beta}\sqrt{2c+1}\begin{pmatrix} a,b,c \\ \alpha,\beta,-(\alpha+\beta)\end{pmatrix}$ and cyclic permutation of the Wigner $3j$symbol. However, since it did not, you must derive the desired CG-coefficient some other way. You could use normalization to determine the remaining coefficient up to a sign, but the question of the sign remains unanswered. To get the the complete answer, you could derive the coefficient directly by applying $\mathbf{J}_{tot-} = \mathbf{J}_{A-} \otimes \mathbf{1}_B + \mathbf{1}_A \otimes \mathbf{J}_{B-}$ repeatedly to both sides of the highest weight state of $j_{tot} = 2$. (In fact, since $j = 0, 1, 2$ are not too large, it would not have been too difficult to derive all the needed coefficients this way.) For example:

$$|2,0\rangle_{tot} = \frac{1}{\sqrt{24}}(\mathbf{J}_{tot-})^2|2,2\rangle_{tot} = \frac{1}{\sqrt{24}}(\mathbf{J}_{A-} \otimes \mathbf{1}_B + \mathbf{1}_A \otimes \mathbf{J}_{B-})^2|1,1\rangle_A \otimes |1,1\rangle_B$$
$$= \frac{1}{\sqrt{24}}(\mathbf{J}_{A-} \otimes \mathbf{1}_B + \mathbf{1}_A \otimes \mathbf{J}_{B-})\left(\sqrt{2}|1,0\rangle_A \otimes |1,1\rangle_B + \sqrt{2}|1,1\rangle_A \otimes |1,0\rangle_B\right)$$
$$= \frac{1}{\sqrt{24}}(2|1,-1\rangle_A \otimes |1,1\rangle_B + 4|1,0\rangle_A \otimes |1,0\rangle_B + 2|1,1\rangle_A \otimes |1,-1\rangle_B)$$
$$= \frac{1}{\sqrt{6}}|1,-1\rangle_A \otimes |1,1\rangle_B + \sqrt{\frac{2}{3}}|1,0\rangle_A \otimes |1,0\rangle_B + \frac{1}{\sqrt{6}}|1,1\rangle_A \otimes |1,-1\rangle_B$$

Gives $C^{2,0}_{1,-1;1,1} = \frac{1}{\sqrt{6}}$, and so on.

(c) $|0,0\rangle_{tot} = \frac{1}{\sqrt{3}}|1,1\rangle_A \otimes |1,-1\rangle_B - \frac{1}{\sqrt{3}}|1,0\rangle_A \otimes |1,0\rangle_B + \frac{1}{\sqrt{3}}|1,-1\rangle_A \otimes |1,1\rangle_B$

$\rho = |0,0\rangle_{tot}\langle 0,0|_{tot}$
$\tilde{\rho}_A = Tr_B[\rho] = \frac{1}{3}(|1,1\rangle_A\langle 1,1|_A + |1,0\rangle_A\langle 1,0|_A + |1,-1\rangle_A\langle 1,-1|_A) = \frac{1}{3}\mathbf{1}_{j_A=1}$
$Tr_A\left[(\tilde{\rho}_A)^2\right] = Tr_A\left[\frac{1}{9}\mathbf{1}_{j_A=1}\right] = \frac{1}{9} \cdot 3 = \frac{1}{3}$
Clearly, $Tr_A\left[(\tilde{\rho}_A)^2\right] \neq 1$

(d) Expressing the state in a coordinate system that is rotated by an angle α about the x-axis is the same as rotating the state by an angle $-\alpha$ in a fixed coordinate system. Essentially what we want to figure out is: $\langle j', m'|U_R|jm\rangle$ for $j = 0,1,2$ where the unitary rotation transformation we want is:
$U_R = \exp\left[-\frac{i}{\hbar}\hat{n}\cdot\vec{J}\phi\right] = \exp\left[\frac{i\alpha}{\hbar}J_x\right]$
Since the j-subspaces are invarient under rotations, i.e. $\langle j', m'|U_R|j,m\rangle = 0$ for $j' \neq j$, I can examine each subspace separately. For each j-subspace that we are interested in, I will compute $J_x^{(j)}$, find the similarity transformation $S^{(j)}$ that diagonalizes $J_x^{(j)}$ (i.e. S is just the

matrix made up of the eigenvectors of J_x in the columns, and if the eigenvectors used are normalized then $S^{-1} = S^\dagger$), and compute the desired elements. (Note: this is equivalent to switching from the $|j,m_z\rangle$ basis to the $|j,m_x\rangle$ basis (in which J_x is diagonal) computing the rotated state, and then switching back to the $|j,m_z\rangle$ basis.)

$J_x^{(0)} = 0$
$\langle 0,0|U_R|0,0\rangle = 1$
$U_R|0,0\rangle = |0,0\rangle$

$$J_x^{(1)} = \frac{1}{2}\left(J_+^{(1)} + J_-^{(1)}\right) = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$S^{(1)} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}$$

$$(S^{(1)})^{-1}J_x^{(1)}S^{(1)} = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\langle 1,m'|U_R|1,0\rangle = \langle 1,m'|S^{(1)}\exp\left[\tfrac{i\alpha}{\hbar}(S^{(1)})^{-1}J_x^{(1)}S^{(1)}\right](S^{(1)})^{-1}|1,0\rangle$$

$$= \langle 1,m'|S^{(1)}\begin{pmatrix} \exp[i\alpha] & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp[-i\alpha] \end{pmatrix}(S^{(1)})^{-1}|1,0\rangle$$

$$= \tfrac{i}{\sqrt{2}}\sin(\alpha)\delta_{m',1} + \cos(\alpha)\delta_{m',0} + \tfrac{i}{\sqrt{2}}\sin(\alpha)\delta_{m',-1}$$

$$U_R|1,0\rangle = i\tfrac{\sqrt{2}}{2}\sin(\alpha)|1,1\rangle + \cos(\alpha)|1,0\rangle + i\tfrac{\sqrt{2}}{2}\sin(\alpha)|1,-1\rangle$$

$$J_x^{(2)} = \tfrac{1}{2}\left(J_+^{(2)} + J_-^{(2)}\right) = \tfrac{\hbar}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$$S^{(2)} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & -\sqrt{\frac{3}{8}} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \sqrt{\frac{3}{8}} & 0 & \frac{1}{2} & 0 & \sqrt{\frac{3}{8}} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\sqrt{\frac{3}{8}} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$(S^{(2)})^{-1}J_x^{(2)}S^{(2)} = \hbar\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

$$\langle 2,m'|U_R|2,m\rangle = \langle 2,m'|S^{(2)}\exp\left[\tfrac{i\alpha}{\hbar}(S^{(2)})^{-1}J_x^{(2)}S^{(2)}\right](S^{(2)})^{-1}|2,m\rangle$$

$$= \langle 2,m'|S^{(2)}\begin{pmatrix} \exp[2i\alpha] & 0 & 0 & 0 & 0 \\ 0 & \exp[i\alpha] & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \exp[-i\alpha] & 0 \\ 0 & 0 & 0 & 0 & \exp[-2i\alpha] \end{pmatrix}(S^{(2)})^{-1}|2,m\rangle$$

$$\langle 2,m'|U_R|2,0\rangle = \left(\begin{array}{c} \sqrt{\tfrac{3}{32}}(\cos(2\alpha)-1)\delta_{m',2} + i\sqrt{\tfrac{3}{8}}\sin(2\alpha)\delta_{m',1} + \left(\tfrac{3}{4}\cos(2\alpha)+\tfrac{1}{4}\right)\delta_{m',0} \\ +i\sqrt{\tfrac{3}{8}}\sin(2\alpha)\delta_{m',-1} + \sqrt{\tfrac{3}{32}}(\cos(2\alpha)-1)\delta_{m',-2} \end{array} \right)$$

$$\langle 2,m'|U_R|2,2\rangle = \left(\begin{array}{c} \left(\tfrac{1}{8}\cos(2\alpha)+\tfrac{1}{2}\cos(\alpha)+\tfrac{3}{8}\right)\delta_{m',2} + i\left(\tfrac{1}{4}\sin(2\alpha)+\tfrac{1}{2}\sin(\alpha)\right)\delta_{m',1} \\ +\sqrt{\tfrac{3}{32}}(\cos(2\alpha)-1)\delta_{m',0} + i\left(\tfrac{1}{4}\sin(2\alpha)-\tfrac{1}{2}\sin(\alpha)\right)\delta_{m',-1} \\ +\left(\tfrac{1}{8}\cos(2\alpha)-\tfrac{1}{2}\cos(\alpha)+\tfrac{3}{8}\right)\delta_{m',-2} \end{array} \right)$$

$$U_R|2,0\rangle = \left(\begin{array}{c} \sqrt{\tfrac{3}{32}}(\cos(2\alpha)-1)|2,2\rangle + i\sqrt{\tfrac{3}{8}}\sin(2\alpha)|2,1\rangle + \left(\tfrac{3}{4}\cos(2\alpha)+\tfrac{1}{4}\right)|2,0\rangle \\ +i\sqrt{\tfrac{3}{8}}\sin(2\alpha)|2,-1\rangle + \sqrt{\tfrac{3}{32}}(\cos(2\alpha)-1)|2,-2\rangle \end{array} \right)$$

$$U_R|2,2\rangle = \left(\begin{array}{c} \left(\tfrac{1}{8}\cos(2\alpha)+\tfrac{1}{2}\cos(\alpha)+\tfrac{3}{8}\right)|2,2\rangle + i\left(\tfrac{1}{4}\sin(2\alpha)+\tfrac{1}{2}\sin(\alpha)\right)|2,1\rangle \\ +\sqrt{\tfrac{3}{32}}(\cos(2\alpha)-1)|2,0\rangle + i\left(\tfrac{1}{4}\sin(2\alpha)-\tfrac{1}{2}\sin(\alpha)\right)|2,-1\rangle \\ +\left(\tfrac{1}{8}\cos(2\alpha)-\tfrac{1}{2}\cos(\alpha)+\tfrac{3}{8}\right)|2,-2\rangle \end{array} \right)$$

Hence, putting it all back together gives:

$$|\Psi'_{AB}\rangle = U_R|\Psi_{AB}\rangle =$$

$$= \frac{1}{\sqrt{2}}\left(\begin{array}{c} \left(\tfrac{1}{8}\cos(2\alpha)+\tfrac{1}{2}\cos(\alpha)+\tfrac{3}{8}\right)|2,2\rangle + i\left(\tfrac{1}{4}\sin(2\alpha)+\tfrac{1}{2}\sin(\alpha)\right)|2,1\rangle \\ +\sqrt{\tfrac{3}{32}}(\cos(2\alpha)-1)|2,0\rangle + i\left(\tfrac{1}{4}\sin(2\alpha)-\tfrac{1}{2}\sin(\alpha)\right)|2,-1\rangle \\ +\left(\tfrac{1}{8}\cos(2\alpha)-\tfrac{1}{2}\cos(\alpha)+\tfrac{3}{8}\right)|2,-2\rangle \end{array} \right)$$

$$+ \frac{1}{\sqrt{12}}\left(\begin{array}{c} \sqrt{\tfrac{3}{32}}(\cos(2\alpha)-1)|2,2\rangle + i\sqrt{\tfrac{3}{8}}\sin(2\alpha)|2,1\rangle + \left(\tfrac{3}{4}\cos(2\alpha)+\tfrac{1}{4}\right)|2,0\rangle \\ +i\sqrt{\tfrac{3}{8}}\sin(2\alpha)|2,-1\rangle + \sqrt{\tfrac{3}{32}}(\cos(2\alpha)-1)|2,-2\rangle \end{array} \right)$$

$$- \frac{1}{\sqrt{4}}\left(i\frac{\sqrt{2}}{2}\sin(\alpha)|1,1\rangle + \cos(\alpha)|1,0\rangle + i\frac{\sqrt{2}}{2}\sin(\alpha)|1,-1\rangle\right)$$

$$+ \frac{1}{\sqrt{6}}|0,0\rangle$$

$$= \left(\begin{array}{c} \left(\tfrac{\sqrt{2}}{8}\cos 2\alpha + \tfrac{\sqrt{2}}{4}\cos\alpha + \tfrac{\sqrt{2}}{8}\right)|2,2\rangle + i\left(\tfrac{\sqrt{2}}{4}\sin 2\alpha + \tfrac{\sqrt{2}}{4}\sin\alpha\right)|2,1\rangle \\ +\left(\tfrac{\sqrt{3}}{4}\cos 2\alpha - \tfrac{\sqrt{3}}{12}\right)|2,0\rangle + i\left(\tfrac{\sqrt{2}}{4}\sin 2\alpha - \tfrac{\sqrt{2}}{4}\sin\alpha\right)|2,-1\rangle \\ +\left(\tfrac{\sqrt{2}}{8}\cos 2\alpha - \tfrac{\sqrt{2}}{4}\cos\alpha + \tfrac{\sqrt{2}}{8}\right)|2,-2\rangle + i\left(-\tfrac{\sqrt{2}}{4}\sin\alpha\right)|1,1\rangle + \left(-\tfrac{1}{2}\cos(\alpha)\right)|1,0\rangle \\ +i\left(-\tfrac{\sqrt{2}}{4}\sin\alpha\right)|1,-1\rangle + \tfrac{1}{\sqrt{6}}|0,0\rangle \end{array} \right)$$

where the $|j,m\rangle$ states are now referring to the basis states of the new coordinate system.