Ph 195a Midterm Solutions:

Problem 1

(a)

1. not Hermitian: $(|0\rangle\langle 1| + i|0\rangle\langle 1|)^\dagger = |1\rangle\langle 0| - i|1\rangle\langle 0| \neq |0\rangle\langle 1| + i|0\rangle\langle 1|$
2. Hermitian: $(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|)^\dagger = |0\rangle\langle 0| + |1\rangle\langle 1| + |3\rangle\langle 2| + |2\rangle\langle 3|$
3. Hermitian: $(a|0\rangle + |1\rangle)^\dagger(a|0\rangle + |1\rangle) = (a^*\langle 0| + \langle 1|)(a|0\rangle + |1\rangle) = |a|^2 + 1$ (real c-numbers are Hermitian)
4. Hermitian: $(a|0\rangle + b^*|1\rangle)^\dagger(b|0\rangle - a^*|1\rangle) = (a^*\langle 0| + b\langle 1|)(b|0\rangle - a^*|1\rangle) = a^*b - ba^* = 0$
   $\Rightarrow (a|0\rangle + b^*|1\rangle)^\dagger(b|0\rangle - a^*|1\rangle)|2\rangle\langle 1| + |3\rangle\langle 3| = |3\rangle\langle 3|$
5. Hermitian: $(|0\rangle\langle 0| + i|1\rangle\langle 0| - i|0\rangle\langle 1| + |1\rangle\langle 1|)^\dagger = |0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|$

(b) The (normalized) eigenvectors of **K** can be found by inspection (or you can find them explicitly if you want):

$|\lambda_0\rangle = |0\rangle$
$|\lambda_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$
$|\lambda_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$
$|\lambda_3\rangle = |3\rangle$

and have eigenvalues: $\lambda_0 = 1, \lambda_1 = 2, \lambda_2 = -2, \lambda_3 = -1$

Hence, $\mathbf{K} = \sum_{i=0}^{3} \lambda_i |\lambda_i\rangle\langle\lambda_i|$

(c) **B** is not a projection operator: $\mathbf{B}^2 = \frac{1}{2}(\mathbf{1} + |\Psi\rangle\langle\Psi|)(\mathbf{1} + |\Psi\rangle\langle\Psi|) = \frac{1}{2}(\mathbf{1} + 3|\Psi\rangle\langle\Psi|) \neq \mathbf{B}$

(d) Clearly $|\Psi\rangle$ is an eigenvector of **B**, and it has eigenvalue $\sqrt{2}$

Hence the spectral decomposition is, $\mathbf{B} = \sqrt{2}|\Psi\rangle\langle\Psi| + \frac{1}{\sqrt{2}}(\mathbf{1} - |\Psi\rangle\langle\Psi|)$ where it is understood that $\mathbf{1} - |\Psi\rangle\langle\Psi|$ can be further decomposed into the sum of three projection operators that correspond to three eigenvectors of **B** that are mutually orthogonal and orthogonal to $|\Psi\rangle$ and which all have eigenvalues of $\frac{1}{\sqrt{2}}$ (i.e. there is some freedom in choosing these eigenvectors since **B** does not provide restraining conditions other than orthogonality with $|\Psi\rangle$)

Problem 2

Since it is factorizable, we can write:

$|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_A\rangle = (a_0|0_A\rangle + a_1|1_A\rangle) \otimes (b_0|0_B\rangle + b_1|1_B\rangle)$ where the coefficients (a_0, a_1, b_0, b_1) are complex numbers.

$\mathbf{U}_{AB}|\Psi_{AB}\rangle = (a_0|0_A\rangle) \otimes (b_0|0_B\rangle) - (a_1|1_A\rangle) \otimes (b_1|1_B\rangle) = a_0 b_0|0_A\rangle \otimes |0_B\rangle - a_1 b_1|1_A\rangle \otimes |1_B\rangle$

This new state is entangled iff both $a_0 b_0$ and $a_1 b_1$ are non-vanishing (i.e. if one of these is zero, then the state is (trivially) factorizable, and if the state is factorizable, then one of these must be zero).

This is the same as the condition that all of a_0, a_1, b_0, b_1 are non-zero, or that $\langle 0_A 0_B|\Psi_{AB}\rangle$ and $\langle 1_A 1_B|\Psi_{AB}\rangle$ are both non-zero.

To see explicitly that this new state is factorizable iff at least one of $a_0 b_0$ or $a_1 b_1$ is zero, consider its purity:

$$\boldsymbol{\rho} = \mathbf{U}_{AB}|\Psi_{AB}\rangle\langle\Psi_{AB}|\mathbf{U}_{AB}^{\dagger}$$
$$\tilde{\boldsymbol{\rho}}_A = \mathbf{Tr}_B[\boldsymbol{\rho}] = |a_0b_0|^2|0_A\rangle\langle 0_A| + |a_1b_1|^2|1_A\rangle\langle 1_A|$$
$$\mathbf{Tr}[\boldsymbol{\rho}] = \mathbf{Tr}_A[\tilde{\boldsymbol{\rho}}_A] = |a_0b_0|^2 + |a_1b_1|^2 = 1$$
$$\tilde{\boldsymbol{\rho}}_A^2 = |a_0b_0|^4|0_A\rangle\langle 0_A| + |a_1b_1|^4|1_A\rangle\langle 1_A|$$
$$\zeta = \mathbf{Tr}_A\left[\tilde{\boldsymbol{\rho}}_A^2\right] = |a_0b_0|^4 + |a_1b_1|^4$$

So if $\zeta = 1$ (i.e. the state is factorizable), then $|a_0b_0|^2 + |a_1b_1|^2 = |a_0b_0|^4 + |a_1b_1|^4$. The only solutions of this equation subject to the constraints $0 \leq \{|a_0b_0|, |a_1b_1|\} \leq 1$ (this follows from assuming $|\Psi_{AB}\rangle$ is normalized, which is assumed when considering ζ) are:

$$|a_0b_0| = 1, |a_1b_1| = 0$$
$$|a_0b_0| = 0, |a_1b_1| = 1$$
$$|a_0b_0| = 0, |a_1b_1| = 0$$

Problem 3

Recall that in the case with a single Hilbert space, the density operator corresponding to a pure state had $\mathbf{Tr}[\boldsymbol{\rho}^2] = 1$, but the density operator of a mixed state had $\mathbf{Tr}[\boldsymbol{\rho}^2] < 1$. This suggests that something similar happens in a joint Hilbert space. Now considering a joint Hilbert space, we know that for a density operator corresponding to a pure state, if the state is factorizable then $\mathbf{Tr}\left[\tilde{\boldsymbol{\rho}}_A^2\right] = 1$, and if the state is entangled then $\mathbf{Tr}[\boldsymbol{\rho}^2] < 1$. Let's see what happens for a mixed state:

$$\begin{aligned}\boldsymbol{\rho} &= \sum_n \Pr(n)|\Psi_n\rangle\langle\Psi_n| = \sum_n \Pr(n)\left(\sum_{jklr} c_{jk}^n c_{lr}^{n*}(|j_A\rangle \otimes |k_B\rangle)(\langle l_A| \otimes \langle r_B|)\right) \\ &= \sum_{\substack{n \\ jklr}} \Pr(n)c_{jk}^n c_{lr}^{n*}|j_A\rangle\langle l_A| \otimes |k_B\rangle\langle r_B|\end{aligned}$$

Where I write the ensemble indices as superscripts and basis state indices as subscripts and segregate the two types of indices underneath summation signs to help distinguish them from each other.

$$\begin{aligned}\tilde{\boldsymbol{\rho}}_A &= \mathbf{Tr}_B[\boldsymbol{\rho}] = \sum_i \langle i_B|\left(\sum_{\substack{n \\ jklr}} \Pr(n)c_{jk}^n c_{lr}^{n*}|j_A\rangle\langle l_A| \otimes |k_B\rangle\langle r_B|\right)|i_B\rangle = \sum_{\substack{n \\ ijklr}} \Pr(n)c_{jk}^n c_{lr}^{n*}|j_A\rangle\langle l_A|\delta_{ik}\delta_{ri} \\ &= \sum_{\substack{n \\ ijl}} \Pr(n)c_{ji}^n c_{li}^{n*}|j_A\rangle\langle l_A|\end{aligned}$$

$$\begin{aligned}\tilde{\boldsymbol{\rho}}_A^2 &= \left(\sum_{\substack{n \\ ijl}} \Pr(n)c_{ji}^n c_{li}^{n*}|j_A\rangle\langle l_A|\right)\left(\sum_{\substack{m \\ qst}} \Pr(m)c_{sq}^m c_{tq}^{m*}|s_A\rangle\langle t_A|\right) = \sum_{\substack{nm \\ ijlqst}} \Pr(n)\Pr(m)c_{ji}^n c_{li}^{n*} c_{sq}^m c_{tq}^{m*}|j_A\rangle\langle t_A|\delta_{ls} \\ &= \sum_{\substack{nm \\ ijqst}} \Pr(n)\Pr(m)c_{ji}^n c_{si}^{n*} c_{sq}^m c_{tq}^{m*}|j_A\rangle\langle t_A|\end{aligned}$$

$$\mathbf{Tr}_A\left[\tilde{\boldsymbol{\rho}}_A^2\right] = \sum_k \langle k_A| \left(\sum_{\substack{nm \\ ijqst}} \Pr(n)\Pr(m) c_{ji}^n c_{si}^{n*} c_{sq}^m c_{tq}^{m*} |j_A\rangle\langle t_A| \right) |k_A\rangle = \sum_{\substack{nm \\ kijqst}} \Pr(n)\Pr(m) c_{ji}^n c_{si}^{n*} c_{sq}^m c_{tq}^{m*} \delta_{kj}\delta_{tk}$$

$$= \sum_{\substack{nm \\ kiqs}} \Pr(n)\Pr(m) c_{ki}^n c_{si}^{n*} c_{sq}^m c_{kq}^{m*}$$

If the ensemble consists of only factorizable states, i.e. $|\Psi_n\rangle = \sum_{jk} a_j^n b_k^n |j_A\rangle \otimes |k_B\rangle$, then $c_{jk}^n = a_j^n b_k^n$ where $\sum_j |a_j^n|^2 = \sum_k |b_k^n|^2 = 1$, so we can write:

$$\mathbf{Tr}_A\left[\tilde{\boldsymbol{\rho}}_A^2\right] = \sum_{\substack{nm \\ kiqs}} \Pr(n)\Pr(m) a_k^n b_i^n a_s^{n*} b_i^{n*} a_s^m b_q^m a_k^{m*} b_q^{m*}$$

$$= \sum_{\substack{nm \\ kiqs}} \Pr(n)\Pr(m) \left(\sum_i b_i^n b_i^{n*} \right) \left(\sum_q b_q^m b_q^{m*} \right) a_k^n a_s^{n*} a_s^m a_k^{m*}$$

$$= \sum_{\substack{nm \\ ks}} \Pr(n)\Pr(m) a_k^n a_s^{n*} a_s^m a_k^{m*}$$

This shows that it is possible to have $\mathbf{Tr}\left[\tilde{\boldsymbol{\rho}}_A^2\right] < 1$ for an ensemble of factorizable states, and so it will not be a good measure of entanglement. For example, consider the ensemble of factorizable states:

$|\Psi_1\rangle = |0_A\rangle \otimes |0_B\rangle$ with $\Pr(1) = \frac{1}{2}$
$|\Psi_2\rangle = |1_A\rangle \otimes |0_B\rangle$ with $\Pr(2) = \frac{1}{2}$
$\mathbf{Tr}_A\left[\tilde{\boldsymbol{\rho}}_A^2\right] = \frac{1}{2}$

(it is interesting to note that this ensemble also satisfies the stronger requirement that the density operator is factorizable, i.e. $\boldsymbol{\rho} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B$, where $\boldsymbol{\rho}_A = \frac{1}{2}(|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|)$ and $\boldsymbol{\rho}_B = |0_B\rangle\langle 0_B|$ and so the same conclusions will hold for mixed states with factorizable density operators)

and the ensemble of entangled states :

$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle)$ with $\Pr(1) = \frac{1}{2}$
$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle \otimes |0_B\rangle - |1_A\rangle \otimes |1_B\rangle)$ with $\Pr(2) = \frac{1}{2}$
$\mathbf{Tr}_A\left[\tilde{\boldsymbol{\rho}}_A^2\right] = \frac{1}{2}$

Clearly, $\mathbf{Tr}_A\left[\tilde{\boldsymbol{\rho}}_A^2\right]$ does not distiguish between these two ensembles at all.

(Any properly described counter-example is considered a sufficient answer for this problem.)

Problem 4

$$\boldsymbol{\rho}_A = \sum_m \Pr(m)|\Psi_m\rangle\langle\Psi_m|$$

$$\boldsymbol{\rho}_B = \sum_n \Pr(n)|\Phi_n\rangle\langle\Phi_n|$$

Where m runs over $1,\ldots,N_A$, where N_A is the number of states $|\Psi_m\rangle$ that compose ensemble A, and n runs over $1,\ldots,N_B$, where N_B is the number of states $|\Phi_n\rangle$ that compose ensemble

B. It is important to recognize that N_A and N_B are not necessarily equal to the dimension of the Hilbert space N_H. I will also assume that $\mathrm{Pr}(m)$ and $\mathrm{Pr}(n)$ are non-zero for all m and n (since if one of these were zero, the corresponding state would not be in the ensemble).
Let $\{|i\rangle\}$ be a basis for the Hilbert space. In this case, i runs over $1,\ldots,N_H$. Then:

$$\begin{aligned}
0 = \mathbf{Tr}[\boldsymbol{\rho}_A\boldsymbol{\rho}_B] &= \sum_i \langle i|\left(\sum_m \mathrm{Pr}(m)|\Psi_m\rangle\langle\Psi_m|\right)\left(\sum_n \mathrm{Pr}(n)|\Phi_n\rangle\langle\Phi_n|\right)|i\rangle \\
&= \sum_{\substack{mn\\ i}} \mathrm{Pr}(m)\,\mathrm{Pr}(n)\langle i|\Psi_m\rangle\langle\Psi_m|\Phi_n\rangle\langle\Phi_n|i\rangle = \sum_{\substack{mn\\ i}} \mathrm{Pr}(m)\,\mathrm{Pr}(n)\langle\Phi_n|i\rangle\langle i|\Psi_m\rangle\langle\Psi_m|\Phi_n\rangle \\
&= \sum_{mn} \mathrm{Pr}(m)\,\mathrm{Pr}(n)\langle\Phi_n|\left(\sum_i |i\rangle\langle i|\right)|\Psi_m\rangle\langle\Psi_m|\Phi_n\rangle = \sum_{mn} \mathrm{Pr}(m)\,\mathrm{Pr}(n)\langle\Phi_n|\Psi_m\rangle\langle\Psi_m|\Phi_n\rangle \\
&= \sum_{mn} \mathrm{Pr}(m)\,\mathrm{Pr}(n)|\langle\Phi_n|\Psi_m\rangle|^2
\end{aligned}$$

Since $\mathrm{Pr}(m)$ and $\mathrm{Pr}(n)$ are non-zero, $\langle\Phi_n|\Psi_m\rangle = 0$ for all m and n. Let $\mathbf{P}_m = |\Psi_m\rangle\langle\Psi_m|$ for some m, so that $\{\mathbf{P}_m, \mathbf{1} - \mathbf{P}_m\}$ is a standard measurement. Then measuring with $\mathbf{P}_m$ gives the results:

$$\begin{aligned}
\mathrm{Pr}\big(m \text{ in } \boldsymbol{\rho}_A\big) = \mathbf{Tr}[\boldsymbol{\rho}_A\mathbf{P}_m] &= \sum_i \langle i|\left(\sum_{m'} \mathrm{Pr}(m')|\Psi_{m'}\rangle\langle\Psi_{m'}|\right)|\Psi_m\rangle\langle\Psi_m|i\rangle \\
&= \sum_i \langle i|\mathrm{Pr}(m)|\Psi_m\rangle\langle\Psi_m|i\rangle = \mathrm{Pr}(m)\sum_i |\langle\Psi_m|i\rangle|^2 > 0
\end{aligned}$$

$$\mathrm{Pr}\big(m \text{ in } \boldsymbol{\rho}_B\big) = \mathbf{Tr}[\boldsymbol{\rho}_B\mathbf{P}_m] = 0$$

So, for a measurement performed using $\mathbf{P}_m$, a non-zero result with indicate ensemble A with absolute certainty. Thus, a standard measurement that can distinguish perfectly between the corresponding mixed-state preparations has been found.
However, we can do better than this. Let $|i\rangle$ be eigenstates of $\boldsymbol{\rho}_B$ with eigenvalues λ_i. In this case, i runs over $1,\ldots,N_H$ since the eigenvectors must span the space. (By symmetry, we could also use the eigenstates of $\boldsymbol{\rho}_A$ and obtain similar results.) First notice:

$$\lambda_i = \langle i|\boldsymbol{\rho}_B|i\rangle = \langle i|\left(\sum_n \mathrm{Pr}(n)|\Phi_n\rangle\langle\Phi_n|\right)|i\rangle = \sum_n \mathrm{Pr}(n)|\langle\Phi_n|i\rangle|^2$$

So that $\lambda_i \geq 0$ for all i, and $\lambda_i = 0$ iff $\langle\Phi_n|i\rangle = 0$ for all n
Replacing $\boldsymbol{\rho}_B$ with its spectral decomposition, and taking the trace over the $\{|i\rangle\}$ basis of the Hilbert space we have:

$$0 = \mathbf{Tr}[\boldsymbol{\rho}_A\boldsymbol{\rho}_B] = \sum_i \langle i|\left(\sum_m \mathrm{Pr}(m)|\Psi_m\rangle\langle\Psi_m|\right)\left(\sum_j \lambda_j|j\rangle\langle j|\right)|i\rangle = \sum_{\substack{m\\ i}} \mathrm{Pr}(m)\lambda_i|\langle\Psi_m|i\rangle|^2$$

So, for each i either $\lambda_i = 0$ or $\langle\Psi_m|i\rangle = 0$ for all m
Thus, we have the condition: $\langle\Psi_m|i\rangle = 0$ for all m or $\langle\Phi_n|i\rangle = 0$ for all n for each eigenstate $|i\rangle$ of $\boldsymbol{\rho}_B$.
Since these eigenstates are mutually orthogonal and span the Hilbert space, we have a complete set of projection operators $\{\mathbf{P}_i = |i\rangle\langle i| : i = 1,\ldots,N_H\}$ where:

$$\mathrm{Pr}\big(i \text{ in } \boldsymbol{\rho}_A\big) = \mathbf{Tr}[\boldsymbol{\rho}_A\mathbf{P}_i] = \sum_j \langle j|\left(\sum_m \mathrm{Pr}(m)|\Psi_m\rangle\langle\Psi_m|\right)\mathbf{P}_i|j\rangle = \sum_m \mathrm{Pr}(m)|\langle\Psi_m|i\rangle|^2$$

$$= \begin{cases} 0 \quad \text{when } \lambda_i \neq 0 \\ 0 \quad \text{when } \lambda_i = 0 \text{ and } \langle \Psi_m | i \rangle = 0 \text{ for all } m \\ \text{non-zero} \quad \text{when } \lambda_i = 0 \text{ and } \langle \Psi_m | i \rangle \neq 0 \text{ for some } m \end{cases}$$

$$\Pr\left(i \text{ in } \boldsymbol{\rho}_B\right) = \mathbf{Tr}[\boldsymbol{\rho}_B \mathbf{P}_i] = \sum_j \langle j | \left(\sum_k \lambda_k |k\rangle\langle k| \right) \mathbf{P}_i |j\rangle = \lambda_i$$

Thus $\{\mathbf{P}_i, \mathbf{1} - \mathbf{P}_i\}$ for any i with non-zero λ_i is also a standard measurement that can distinguish perfectly between the corresponding mixed-state preparations. But, noticing that:

$$\sum_i \lambda_i = \sum_i \langle i | \boldsymbol{\rho}_B | i \rangle = \mathbf{Tr}[\boldsymbol{\rho}_B] = 1$$

If we define $\mathbf{P}_B = \sum_{\{i:\lambda_i \neq 0\}} \mathbf{P}_i$ (i.e. the projection operator equal to the sum of the $\mathbf{P}_i$ corresponding to non-zero eigenvalues λ_i) and $\mathbf{P}_A = \mathbf{1} - \mathbf{P}_B$, then $\{\mathbf{P}_A, \mathbf{P}_B\}$ is a standard measurement, and:

$$\Pr\left(A \text{ in } \boldsymbol{\rho}_A\right) = \mathbf{Tr}[\boldsymbol{\rho}_A \mathbf{P}_A] = \mathbf{Tr}[\boldsymbol{\rho}_A(\mathbf{1} - \mathbf{P}_B)] = \mathbf{Tr}[\boldsymbol{\rho}_A] - \mathbf{Tr}[\boldsymbol{\rho}_A \mathbf{P}_B] = 1 - 0 = 1$$

$$\Pr\left(B \text{ in } \boldsymbol{\rho}_A\right) = \mathbf{Tr}[\boldsymbol{\rho}_A \mathbf{P}_B] = \mathbf{Tr}[\boldsymbol{\rho}_A \mathbf{P}_B] = 0$$

$$\Pr\left(A \text{ in } \boldsymbol{\rho}_B\right) = \mathbf{Tr}[\boldsymbol{\rho}_B \mathbf{P}_A] = \mathbf{Tr}[\boldsymbol{\rho}_B(\mathbf{1} - \mathbf{P}_B)] = \mathbf{Tr}[\boldsymbol{\rho}_A] - \mathbf{Tr}[\boldsymbol{\rho}_A \mathbf{P}_B] = 1 - 1 = 0$$

$$\Pr\left(B \text{ in } \boldsymbol{\rho}_B\right) = \mathbf{Tr}[\boldsymbol{\rho}_B \mathbf{P}_B] = 1$$

Thus, a standard measurement that distinguishes perfectly between the corresponding mixed-state preparations for every measurement performed can also been found.

Problem 5

(a) $\boldsymbol{\rho} = |\Psi_{ABC}\rangle\langle\Psi_{ABC}|$

$$= \tfrac{1}{2}\left(|0_A 0_B 0_C\rangle\langle 0_A 0_B 0_C| + |0_A 0_B 0_C\rangle\langle 1_A 1_B 1_C| + |1_A 1_B 1_C\rangle\langle 0_A 0_B 0_C| + |1_A 1_B 1_C\rangle\langle 1_A 1_B 1_C|\right)$$

$$\tilde{\boldsymbol{\rho}}_{AB} = \mathbf{Tr}_C[\boldsymbol{\rho}] = \sum_{k=0,1} \langle k_C | \boldsymbol{\rho} | k_C \rangle = \tfrac{1}{2}\left(|0_A 0_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B|\right)$$

(b) Let $\mathbf{P}_0 = \mathbf{1}_A \otimes \mathbf{1}_B \otimes |0_C\rangle\langle 0_C|$ and $\mathbf{P}_1 = \mathbf{1}_A \otimes \mathbf{1}_B \otimes |1_C\rangle\langle 1_C|$

$$\Pr(0) = \mathbf{Tr}[\boldsymbol{\rho}\mathbf{P}_0] = \mathbf{Tr}\left[\tfrac{1}{2}\left(|0_A 0_B 0_C\rangle\langle 0_A 0_B 0_C| + |0_A 0_B 0_C\rangle\langle 1_A 1_B 1_C|\right)\right] = \tfrac{1}{2}$$

$$\Pr(1) = \mathbf{Tr}[\boldsymbol{\rho}\mathbf{P}_1] = \mathbf{Tr}\left[\tfrac{1}{2}\left(|1_A 1_B 1_C\rangle\langle 0_A 0_B 0_C| + |1_A 1_B 1_C\rangle\langle 1_A 1_B 1_C|\right)\right] = \tfrac{1}{2}$$

With post-measurement states:

$$|\Psi_0\rangle = \frac{\mathbf{P}_0|\Psi_{ABC}\rangle}{\sqrt{\langle\Psi_{ABC}|\mathbf{P}_0|\Psi_{ABC}\rangle}} = \frac{\mathbf{P}_0|\Psi_{ABC}\rangle}{\sqrt{\mathbf{Tr}[\rho\mathbf{P}_0]}} = |0_A 0_B 0_C\rangle$$

$$|\Psi_1\rangle = \frac{\mathbf{P}_1|\Psi_{ABC}\rangle}{\sqrt{\langle\Psi_{ABC}|\mathbf{P}_1|\Psi_{ABC}\rangle}} = \frac{\mathbf{P}_1|\Psi_{ABC}\rangle}{\sqrt{\mathbf{Tr}[\rho\mathbf{P}_1]}} = |1_A 1_B 1_C\rangle$$

This is consistent with the result from (a) since

$$\boldsymbol{\rho}^{01} = \Pr(0)|\Psi_0\rangle\langle\Psi_0| + \Pr(1)|\Psi_1\rangle\langle\Psi_1| = \tfrac{1}{2}\left(|0_A 0_B 0_C\rangle\langle 0_A 0_B 0_C| + |1_A 1_B 1_C\rangle\langle 1_A 1_B 1_C|\right)$$

$$\tilde{\boldsymbol{\rho}}^{01}_{AB} = \mathbf{Tr}_C[\boldsymbol{\rho}^{pm}] = \tfrac{1}{2}\left(|0_A 0_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B|\right) = \tilde{\boldsymbol{\rho}}_{AB}$$

(c) Let $\mathbf{P}_x = \mathbf{1}_A \otimes \mathbf{1}_B \otimes |x_C\rangle\langle x_C|$ and $\mathbf{P}_y = \mathbf{1}_A \otimes \mathbf{1}_B \otimes |y_C\rangle\langle y_C|$

$$\Pr(x) = \mathbf{Tr}[\boldsymbol{\rho}\mathbf{P}_x] = \tfrac{1}{2}$$

$$\Pr(y) = \mathbf{Tr}\left[\boldsymbol{\rho}\mathbf{P}_y\right] = \tfrac{1}{2}$$

With post-measurement states:

$$
\begin{aligned}
|\Psi_x\rangle &= \frac{\mathbf{P}_x|\Psi_{ABC}\rangle}{\sqrt{\langle\Psi_{ABC}|\mathbf{P}_x|\Psi_{ABC}\rangle}} = \frac{\mathbf{P}_x|\Psi_{ABC}\rangle}{\sqrt{\mathbf{Tr}[\rho\mathbf{P}_x]}} = \tfrac{1}{2}(|0_A0_B0_C\rangle + |0_A0_B1_C\rangle + |1_A1_B0_C\rangle + |1_A1_B1_C\rangle) \\
&= \tfrac{1}{\sqrt{2}}(|0_A0_B\rangle + |1_A1_B\rangle) \otimes |x_C\rangle \\
|\Psi_y\rangle &= \frac{\mathbf{P}_y|\Psi_{ABC}\rangle}{\sqrt{\langle\Psi_{ABC}|\mathbf{P}_y|\Psi_{ABC}\rangle}} = \frac{\mathbf{P}_y|\Psi_{ABC}\rangle}{\sqrt{\mathbf{Tr}[\rho\mathbf{P}_y]}} = \tfrac{1}{2}(|0_A0_B0_C\rangle - |0_A0_B1_C\rangle - |1_A1_B0_C\rangle + |1_A1_B1_C\rangle) \\
&= \tfrac{1}{\sqrt{2}}(|0_A0_B\rangle - |1_A1_B\rangle) \otimes |y_C\rangle
\end{aligned}
$$

This is consistent with the result from (a) since

$$
\begin{aligned}
\boldsymbol{\rho}^{xy} &= \Pr(x)|\Psi_x\rangle\langle\Psi_x| + \Pr(y)|\Psi_y\rangle\langle\Psi_y| \\
&= \tfrac{1}{4}((|0_A0_B\rangle + |1_A1_B\rangle)(\langle 0_A0_B| + \langle 1_A1_B|) \otimes |x_C\rangle\langle x_C|) + \tfrac{1}{4}((|0_A0_B\rangle - |1_A1_B\rangle)(\langle 0_A0_B| - \langle 1_A1_B|) \otimes \\
\tilde{\boldsymbol{\rho}}^{xy}_{AB} &= \mathbf{Tr}_C[\boldsymbol{\rho}^{pm}] = \langle x_C|\boldsymbol{\rho}^{pm}|x_C\rangle + \langle y_C|\boldsymbol{\rho}^{pm}|y_C\rangle \\
&= \tfrac{1}{4}(|0_A0_B\rangle + |1_A1_B\rangle)(\langle 0_A0_B| + \langle 1_A1_B|) + \tfrac{1}{4}(|0_A0_B\rangle - |1_A1_B\rangle)(\langle 0_A0_B| - \langle 1_A1_B|) \\
&= \tfrac{1}{2}(|0_A0_B\rangle\langle 0_A0_B| + |1_A1_B\rangle\langle 1_A1_B|) = \tilde{\boldsymbol{\rho}}_{AB}
\end{aligned}
$$

(d) Assuming equal probabilities for Charlie's two measurement bases, the ensemble on $H_A \otimes H_B$ will be:

$|0_{AB}\rangle = |0_A0_B\rangle$ with $\Pr(0_{AB}) = \frac{1}{4}$

$|1_{AB}\rangle = |1_A1_B\rangle$ with $\Pr(1_{AB}) = \frac{1}{4}$

$|x_{AB}\rangle = \frac{1}{\sqrt{2}}(|0_A0_B\rangle + |1_A1_B\rangle)$ with $\Pr(x_{AB}) = \frac{1}{4}$

$|y_{AB}\rangle = \frac{1}{\sqrt{2}}(|0_A0_B\rangle - |1_A1_B\rangle)$ with $\Pr(y_{AB}) = \frac{1}{4}$

$$
\begin{aligned}
\boldsymbol{\rho}^{pm}_{AB} &= \tfrac{1}{4}(|0_{AB}\rangle\langle 0_{AB}| + |1_{AB}\rangle\langle 1_{AB}| + |x_{AB}\rangle\langle x_{AB}| + |y_{AB}\rangle\langle y_{AB}|) \\
&= \tfrac{1}{2}(|0_A0_B\rangle\langle 0_A0_B| + |1_A1_B\rangle\langle 1_A1_B|) = \tilde{\boldsymbol{\rho}}_{AB}
\end{aligned}
$$

We can also use the results from (b) and (c) to write the post-measurement density operator on $H_A \otimes H_B \otimes H_c$ and its reduced density operator on $H_A \otimes H_B$:

$$
\begin{aligned}
\boldsymbol{\rho}^{pm} &= \Pr(01 - basis)\boldsymbol{\rho}^{01} + \Pr(xy - basis)\boldsymbol{\rho}^{xy} = \tfrac{1}{2}\boldsymbol{\rho}^{01} + \tfrac{1}{2}\boldsymbol{\rho}^{xy} \\
\tilde{\boldsymbol{\rho}}^{xy}_{AB} &= \mathbf{Tr}_C[\boldsymbol{\rho}^{pm}] = \tfrac{1}{2}(\langle 0_C|\boldsymbol{\rho}^{01}|0_C\rangle + \langle 1_C|\boldsymbol{\rho}^{01}|1_C\rangle + \langle x_C|\boldsymbol{\rho}^{xy}|x_C\rangle + \langle y_C|\boldsymbol{\rho}^{xy}|y_C\rangle) \\
&= \tfrac{1}{2}(\tilde{\boldsymbol{\rho}}_{AB} + \tilde{\boldsymbol{\rho}}_{AB}) = \tilde{\boldsymbol{\rho}}_{AB}
\end{aligned}
$$

So these are all consistent with the result from (a).

If Charlie informs Alice and Bob of which measurement he performed and the outcome, then the post-measurement state on $H_A \otimes H_B$ is completely determined, so Alice and Bob should change their ensemble description to be probability of 1 in the unique state. Specifically:

If Charlie measured $|0_C\rangle$ in the 01-basis, then: $|0_{AB}\rangle$ with $\Pr(0_{AB}) = 1$

If Charlie measured $|1_C\rangle$ in the 01-basis, then: $|1_{AB}\rangle$ with $\Pr(1_{AB}) = 1$

If Charlie measured $|x_C\rangle$ in the xy-basis, then: $|x_{AB}\rangle$ with $\Pr(x_{AB}) = 1$

If Charlie measured $|y_C\rangle$ in the xy-basis, then: $|y_{AB}\rangle$ with $\Pr(y_{AB}) = 1$