Ph 195b Midterm Solutions:

Problem 1

(a) Note that $\mathbf{S}(\xi) = \exp[-\lambda \mathbf{A}]$ where $\lambda = -\frac{1}{2}$ and $\mathbf{A} = \xi^* \mathbf{a}^2 - \xi(\mathbf{a}^\dagger)^2$ gives $\mathbf{S}^\dagger(\xi) = \exp[\lambda \mathbf{A}]$, so then all we need to do is figure out which relation applies to this case (i.e. figure out the commutation relation of $\mathbf{A}$ with $\mathbf{a}$).

$[\mathbf{A}, \mathbf{a}] = \left[\xi^* \mathbf{a}^2 - \xi(\mathbf{a}^\dagger)^2, \mathbf{a}\right] = \left[-\xi(\mathbf{a}^\dagger)^2, \mathbf{a}\right] = -\xi(\mathbf{a}^\dagger[\mathbf{a}^\dagger, \mathbf{a}] + [\mathbf{a}^\dagger, \mathbf{a}]\mathbf{a}^\dagger) = 2\xi \mathbf{a}^\dagger$

$[\mathbf{A}, [\mathbf{A}, \mathbf{a}]] = \left[\xi^* \mathbf{a}^2 - \xi(\mathbf{a}^\dagger)^2, 2\xi \mathbf{a}^\dagger\right] = 2\xi^* \xi[\mathbf{a}^2, \mathbf{a}^\dagger] = 2|\xi|^2(\mathbf{a}[\mathbf{a}, \mathbf{a}^\dagger] + \mathbf{a}[\mathbf{a}, \mathbf{a}^\dagger]) = 4|\xi|^2 \mathbf{a}$

So we can apply relation 3 for: $[\mathbf{A}, [\mathbf{A}, \mathbf{a}]] = \beta \mathbf{a}$ with $\beta = 4|\xi|^2$, giving:

$\mathbf{S}^\dagger(\xi)\mathbf{a}\mathbf{S}(\xi) = \mathbf{a}\cosh\left[-\frac{1}{2}\sqrt{4|\xi|^2}\right] + \frac{2\xi \mathbf{a}^\dagger}{\sqrt{4|\xi|^2}} \sinh\left[-\frac{1}{2}\sqrt{4|\xi|^2}\right] = \mathbf{a}\cosh[|\xi|] - \mathbf{a}^\dagger \frac{\xi}{|\xi|} \sinh[|\xi|]$

where I used $\cosh[-x] = \cosh[x]$ and $\sinh[-x] = -\sinh[x]$. Then letting $\xi = r\exp[2i\varphi]$, we get:

$\mathbf{S}^\dagger(\xi)\mathbf{a}\mathbf{S}(\xi) = \mathbf{a}\cosh[r] - \mathbf{a}^\dagger \exp[2i\varphi]\sinh[r]$

Taking the Hermitian conjugate gives:

$\mathbf{S}^\dagger(\xi)\mathbf{a}^\dagger\mathbf{S}(\xi) = \mathbf{a}^\dagger \cosh[r] - \mathbf{a}\exp[-2i\varphi]\sinh[r]$

(b) Before actually solving this problem, I want to say a few things that will (hopefully) give a clearer understanding of the whole squeezing business. Since $\mathbf{S}(\xi) = \exp\left[\frac{1}{2}\xi^* \mathbf{a}^2 - \frac{1}{2}\xi(\mathbf{a}^\dagger)^2\right]$ is a unitary operator, we can use it as the transformation operator in a unitary transformation:

$|\Psi\rangle \rightarrow |\Psi'\rangle = \mathbf{S}^\dagger(\xi)|\Psi\rangle$

$\mathbf{L} \rightarrow \mathbf{L}' = \mathbf{S}^\dagger(\xi)\mathbf{L}\mathbf{S}(\xi)$

In particular, this gives:

$|n'\rangle = \mathbf{S}^\dagger(\xi)|n\rangle$

$\mathbf{a}' = \mathbf{S}^\dagger(\xi)\mathbf{a}\mathbf{S}(\xi)$

with the eigenspectra of all operators unchanged under transformation (recall the discussion on unitary transformations). What that means is that the transformed number states are actually number states of a new system which has annihilation, creation, number, etc. operators that are exactly the corresponding transformed operators (i.e. $\mathbf{a}'|n'\rangle = \sqrt{n}\,|(n-1)'\rangle, \mathbf{a}'^\dagger|n'\rangle = \sqrt{n+1}\,|(n+1)'\rangle, \mathbf{N}'|n'\rangle = n|n'\rangle$, etc.), so the transformed system is simply another harmonic oscillator-type system. While this result is expected from any unitary transformation, what is special about the squeezing operator is that it provides a unitary transformation that is perfectly designed (as you will see below) so that the new system corresponds to a harmonic oscillator with a different oscillation frequency. With that said, the sudden switch of potentials $\mathbf{V} \rightarrow \mathbf{V}'$ can be thought of as an easy way to produce an eigenstate of an ω-oscillator in an ω'-oscillator system, and the sqeezing operator tells us how represent this mathematically. Another property worth mentioning is that $\mathbf{S}(-\xi) = \mathbf{S}^\dagger(\xi) = \mathbf{S}^{-1}(\xi)$, so that sqeezing with $-\xi$ is exactly the reverse of sqeezing with ξ (which is really just saying that ξ is a good label for the sqeezing operator... remember in class we thought it was a strange label, since $\sqrt{\xi}$ was seen to be the physically meaningful quantity under time-evolution). This also makes sense in light of the notation $\xi = r\exp[2i\varphi]$ where the angle φ corresponds to the sqeezing axis (in the complex α plane) and $-\xi$ is just a shift in φ of $\frac{\pi}{2}$ (i.e. a $-\xi$ sqeeze is perpendicular to a ξ sqeeze).

So now it is clear that the idea behind this problem is this: For harmonic oscillator potentials $\mathbf{V}(\mathbf{x}) = \frac{1}{2}m\omega^2\mathbf{x}^2$ and $\mathbf{V}'(\mathbf{x}) = \frac{1}{2}m(\omega')^2\mathbf{x}^2$, find the value of ξ that makes $\mathbf{S}(\xi)$ a transformation between states and operators of the two systems. To do this, we let **a** be the annihilation operator for the ω-oscillator and **b** be the annihilation operator for the ω'-oscillator, and solve for $\mathbf{b} = \mathbf{a}'$:

$$\mathbf{b} = \mathbf{S}^\dagger(\xi)\mathbf{a}\mathbf{S}(\xi) = \mathbf{a}\cosh[r] - \mathbf{a}^\dagger\exp[2i\varphi]\sinh[r]$$

$$\mathbf{b}^\dagger = \mathbf{S}^\dagger(\xi)\mathbf{a}^\dagger\mathbf{S}(\xi) = \mathbf{a}^\dagger\cosh[r] - \mathbf{a}\exp[-2i\varphi]\sinh[r]$$

Then noting that:

$$\mathbf{x} = \sqrt{\tfrac{\hbar}{2m\omega}}(\mathbf{a}+\mathbf{a}^\dagger) = \sqrt{\tfrac{\hbar}{2m\omega'}}(\mathbf{b}+\mathbf{b}^\dagger)$$

$$\mathbf{p} = -i\sqrt{\tfrac{m\hbar\omega}{2}}(\mathbf{a}-\mathbf{a}^\dagger) = -i\sqrt{\tfrac{m\hbar\omega'}{2}}(\mathbf{b}-\mathbf{b}^\dagger)$$

we get:

$$\mathbf{b} = \sqrt{\tfrac{m\omega'}{2\hbar}}\left(\mathbf{x}+i\frac{\mathbf{p}}{m\omega'}\right) = \sqrt{\tfrac{m\omega'}{2\hbar}}\left(\sqrt{\tfrac{\hbar}{2m\omega}}(\mathbf{a}+\mathbf{a}^\dagger)+\frac{1}{m\omega'}\sqrt{\tfrac{m\hbar\omega}{2}}(\mathbf{a}-\mathbf{a}^\dagger)\right)$$

$$= \frac{1}{2}\left(\sqrt{\tfrac{\omega'}{\omega}}+\sqrt{\tfrac{\omega}{\omega'}}\right)\mathbf{a} + \frac{1}{2}\left(\sqrt{\tfrac{\omega'}{\omega}}-\sqrt{\tfrac{\omega}{\omega'}}\right)\mathbf{a}^\dagger$$

$$\mathbf{b}^\dagger = \frac{1}{2}\left(\sqrt{\tfrac{\omega'}{\omega}}+\sqrt{\tfrac{\omega}{\omega'}}\right)\mathbf{a}^\dagger + \frac{1}{2}\left(\sqrt{\tfrac{\omega'}{\omega}}-\sqrt{\tfrac{\omega}{\omega'}}\right)\mathbf{a}$$

Matching these to the values above, we get the results:

$$\cosh[r] = \frac{1}{2}\left(\sqrt{\tfrac{\omega'}{\omega}}+\sqrt{\tfrac{\omega}{\omega'}}\right)$$

$$\exp[2i\varphi] = \exp[-2i\varphi] = -sign\left(\sqrt{\tfrac{\omega'}{\omega}}-\sqrt{\tfrac{\omega}{\omega'}}\right)$$

$$\sinh[r] = \frac{1}{2}\left|\sqrt{\tfrac{\omega'}{\omega}}-\sqrt{\tfrac{\omega}{\omega'}}\right|$$

(from which it should be clear that the sqeezing operator was perfectly designed, as claimed.) These equations reduce to:

$$r = \frac{1}{2}\left|\log\left(\frac{\omega}{\omega'}\right)\right|$$

$$\exp[2i\varphi] = sign(\omega-\omega')$$

$$\xi = \frac{1}{2}\log\left(\frac{\omega}{\omega'}\right)$$

Problem 2

First find the exact form of $\mathbf{D}(\beta)\mathbf{D}(\alpha)$, by noting that:

$$[\beta\mathbf{a}^\dagger-\beta^*\mathbf{a},\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}] = -\alpha^*\beta[\mathbf{a}^\dagger,\mathbf{a}]-\beta^*\alpha[\mathbf{a},\mathbf{a}^\dagger] = (\alpha^*\beta-\alpha\beta^*)\mathbf{1}$$

which commutes with both $\beta\mathbf{a}^\dagger-\beta^*\mathbf{a}$ and $\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}$, so using relation 1, we get:

$$\mathbf{D}(\beta)\mathbf{D}(\alpha) = \exp[\beta\mathbf{a}^\dagger-\beta^*\mathbf{a}]\exp[\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}] = \exp\left[\beta\mathbf{a}^\dagger-\beta^*\mathbf{a}+\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}+\tfrac{1}{2}(\alpha^*\beta-\alpha\beta^*)\mathbf{1}\right]$$

$$= \exp\left[\tfrac{1}{2}(\alpha^*\beta-\alpha\beta^*)\right]\exp[(\beta+\alpha)\mathbf{a}^\dagger-(\beta^*+\alpha^*)\mathbf{a}] = \exp\left[\tfrac{1}{2}(\alpha^*\beta-\alpha\beta^*)\right]\mathbf{D}(\beta+\alpha)$$

Using the results of Problem 1(a) with $r = \xi, \varphi = 0$:

$$\mathbf{S}^\dagger(\xi)(\gamma\mathbf{a}^\dagger-\gamma^*\mathbf{a})\mathbf{S}(\xi)$$

$$= \gamma(\mathbf{a}^\dagger\cosh[\xi]-\mathbf{a}\sinh[\xi])-\gamma^*(\mathbf{a}\cosh[\xi]-\mathbf{a}^\dagger\sinh[\xi])$$

$$= \mathbf{a}^\dagger(\gamma\cosh[\xi]+\gamma^*\sinh[\xi])-\mathbf{a}(\gamma\sinh[\xi]+\gamma^*\cosh[\xi])$$

$$= \eta\mathbf{a}^\dagger-\eta^*\mathbf{a}$$

where $\eta = \gamma\cosh[\xi]+\gamma^*\sinh[\xi]$. This gives:

$\mathbf{S}^\dagger(\xi)\mathbf{D}(\gamma)\mathbf{S}(\xi) = \mathbf{D}(\eta)$
So putting it all together with $\gamma = \alpha_2 - \alpha_1$, we get:

$$\begin{aligned}\langle\alpha_1;\xi|\alpha_2;\xi\rangle &= \langle 0|\mathbf{S}^\dagger(\xi)\mathbf{D}^\dagger(\alpha_1)\mathbf{D}(\alpha_2)\mathbf{S}(\xi)|0\rangle = \langle 0|\mathbf{S}^\dagger(\xi)\mathbf{D}(-\alpha_1)\mathbf{D}(\alpha_2)\mathbf{S}(\xi)|0\rangle \\ &= \langle 0|\mathbf{S}^\dagger(\xi)\exp\left[\tfrac{1}{2}(-\alpha_2^*\alpha_1 + \alpha_2\alpha_1^*)\right]\mathbf{D}(\alpha_2-\alpha_1)\mathbf{S}(\xi)|0\rangle \\ &= \exp\left[\tfrac{1}{2}(-\alpha_2^*\alpha_1 + \alpha_2\alpha_1^*)\right]\langle 0|\mathbf{S}^\dagger(\xi)\mathbf{D}(\gamma)\mathbf{S}(\xi)|0\rangle \\ &= \exp\left[\tfrac{1}{2}(-\alpha_2^*\alpha_1 + \alpha_2\alpha_1^*)\right]\langle 0|\mathbf{D}(\eta)|0\rangle = \exp\left[\tfrac{1}{2}(-\alpha_2^*\alpha_1 + \alpha_2\alpha_1^*)\right]\langle 0|\eta\rangle \\ &= \exp\left[\tfrac{1}{2}(-\alpha_2^*\alpha_1 + \alpha_2\alpha_1^*)\right]\exp\left[-\tfrac{1}{2}|\eta|^2\right] = \exp\left[\tfrac{1}{2}\left(-\alpha_2^*\alpha_1 + \alpha_2\alpha_1^* - |\eta|^2\right)\right] \\ &= \exp\left[\tfrac{1}{2}\left(-\alpha_2^*\alpha_1 + \alpha_2\alpha_1^* - |(\alpha_2-\alpha_1)\cosh[\xi] + (\alpha_2-\alpha_1)^*\sinh[\xi]|^2\right)\right]\end{aligned}$$

When α_1 and α_2 are pure real or pure imaginary, $(-\alpha_2^*\alpha_1 + \alpha_2\alpha_1^*) = 0$.
When α_1 and α_2 are purely real, we have:

$$|\eta|^2 = |(\alpha_2-\alpha_1)\cosh[\xi] + (\alpha_2-\alpha_1)^*\sinh[\xi]|^2 = |\alpha_2-\alpha_1|^2\exp[2\xi]$$
$$\langle\alpha_1;\xi|\alpha_2;\xi\rangle = \exp\left[-\tfrac{1}{2}|\alpha_2-\alpha_1|^2\exp[2\xi]\right]$$

and when α_1 and α_2 are purely imaginary, we have:

$$|\eta|^2 = |(\alpha_2-\alpha_1)\cosh[\xi] + (\alpha_2-\alpha_1)^*\sinh[\xi]|^2 = |\alpha_2-\alpha_1|^2\exp[-2\xi]$$
$$\langle\alpha_1;\xi|\alpha_2;\xi\rangle = \exp\left[-\tfrac{1}{2}|\alpha_2-\alpha_1|^2\exp[-2\xi]\right]$$

Thus, we see that the inner product of the two states is smaller when the states lie on the axis (in the complex α plane) of the sqeezing, and are larger when they are on the axis perpendicular to the sqeezing axis.

Problem 3

Let $\omega_+ = \sqrt{\frac{k_+}{m}}$ and $\omega_- = \sqrt{\frac{k_-}{m}}$ as usual. I will solve this problem two ways: brute force and finesse.

First the brute force route:

Let **a** be the annihilation operator of $\mathbf{H}_+$, **b** be the annihilation operator of $\mathbf{H}_-$, and $\mathbf{S}(\xi)$ with $\xi = \frac{1}{2}\log(\frac{\omega_+}{\omega_-})$ be the sqeezing operator that gives the unitary transformation from the $\mathbf{H}_+$ system to the $\mathbf{H}_-$ system (as described in Problem 1(b)). In this problem, I will label the states and operators (other than the annihilation/creation operators) of the $\mathbf{H}_+$ and $\mathbf{H}_-$ systems with a subscript $+$ and $-$ respectively. Operators are often defined in a basis dependent manner, so it is important to specify what basis an operator is acting on, because in general the transformation of operators are non-trivial. For example, $\mathbf{D}_+(\alpha) = \exp[\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a}]$ and $\mathbf{D}_-(\alpha) = \exp[\alpha\mathbf{b}^\dagger - \alpha^*\mathbf{b}]$. The tranformations of the annihilation and creation operators: $\mathbf{b} = \mathbf{S}^\dagger(\xi)\mathbf{a}\mathbf{S}(\xi)$ and $\mathbf{b}^\dagger = \mathbf{S}^\dagger(\xi)\mathbf{a}^\dagger\mathbf{S}(\xi)$, give certain operators similarly nice transformation properties; in particular, the displacement operator is one such operator, and its transformation will be useful later, so I will show it here:

$$\begin{aligned}\mathbf{D}'_+(\alpha) &= \mathbf{S}^\dagger(\xi)\mathbf{D}_+(\boldsymbol{\alpha})\mathbf{S}(\xi) = \mathbf{S}^\dagger(\xi)\exp[\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a}]\mathbf{S}(\xi) = \mathbf{S}^\dagger(\xi)\left[\sum_{k=0}^{\infty}\tfrac{1}{k!}(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a})^k\right]\mathbf{S}(\xi) \\ &= \sum_{k=0}^{\infty}\tfrac{1}{k!}\mathbf{S}^\dagger(\xi)(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a})^k\mathbf{S}(\xi) = \sum_{k=0}^{\infty}\tfrac{1}{k!}(\mathbf{S}^\dagger(\xi)(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a})\mathbf{S}(\xi))^k = \sum_{k=0}^{\infty}\tfrac{1}{k!}(\alpha\mathbf{b}^\dagger - \alpha^*\mathbf{b})^k \\ &= \exp[\alpha\mathbf{b}^\dagger - \alpha^*\mathbf{b}] = \mathbf{D}_-(\alpha)\end{aligned}$$

So, in general the displacement α in one basis transforms to a displacement α in another basis. In fact, if this were not the case, then there would be cause for alarm... remember coherent states are eigenstates of the LHO, and an eigenstate should retain its

eigenspectrum under a unitary transformation (i.e. $\mathbf{a}|\alpha_+\rangle = \alpha|\alpha_+\rangle \Rightarrow \mathbf{b}|\alpha_-\rangle = \mathbf{a}'|\alpha'_+\rangle = \alpha|\alpha'_+\rangle = \alpha|\alpha_-\rangle$). Be careful however, since in general $\mathbf{L}'_+ = \mathbf{S}^\dagger(\xi)\mathbf{L}_+\mathbf{S}(\xi) \neq \mathbf{L}_-$, for example:
$\mathbf{H}'_+ = \mathbf{S}^\dagger(\xi)\mathbf{H}_+\mathbf{S}(\xi) = \mathbf{S}^\dagger(\xi)\hbar\omega_+\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)\mathbf{S}(\xi) = \hbar\omega_+\left(\mathbf{b}^\dagger\mathbf{b} + \frac{1}{2}\right) \neq \mathbf{H}_-$. Of course, since $\mathbf{S}(\xi)$ is the transformation function, it transforms trivially (i.e. $\mathbf{S}'_+(\xi) = \mathbf{S}^\dagger_+(\xi)\mathbf{S}_+(\xi)\mathbf{S}_+(\xi) = \mathbf{S}_+(\xi)$), and since $\mathbf{S}_-(\xi) = \mathbf{S}'_+(\xi)$ by the same reasoning that gave $\mathbf{D}'_+(\alpha) = \mathbf{D}_-(\alpha)$, we do not need to distinguish between $\mathbf{S}_+(\xi)$ and $\mathbf{S}_-(\xi)$ and can unambiguously drop the subscript labeling what basis it acts on (justifying why I did not label it in the first place).

Now I begin solving the problem by noting that:

$$\mathbf{T}(t,0) = \exp\left[-i\tfrac{t}{\hbar}\mathbf{H}\right] = \exp\left[-i\tfrac{t}{\hbar}\begin{pmatrix}\mathbf{H}_+ & 0\\ 0 & \mathbf{H}_-\end{pmatrix}\right] = \begin{pmatrix}\exp\left[-i\tfrac{t}{\hbar}\mathbf{H}_+\right] & 0\\ 0 & \exp\left[-i\tfrac{t}{\hbar}\mathbf{H}_-\right]\end{pmatrix}$$
$$= \begin{pmatrix}\mathbf{T}_+(t,0) & 0\\ 0 & \mathbf{T}_-(t,0)\end{pmatrix}$$

where $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ and $\mathbf{T}_+(t,0)$ and $\mathbf{T}_-(t,0)$ are the time evolution operators for $\mathbf{H}_+$ and $\mathbf{H}_-$ respectively.

To save myself the trouble of time evolving coherent squeezed states, I will switch to the Heisenberg picture:

$$|\overline{\Psi}(t)\rangle = |\Psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{D}_+(\alpha)|0_+\rangle\\ \mathbf{D}_+(\alpha)|0_+\rangle\end{pmatrix}$$

$$\overline{\mathbf{x}} = \mathbf{T}(0,t)\mathbf{x}\mathbf{T}(t,0) = \mathbf{T}(0,t)(\mathbf{1}\otimes\mathbf{x})\mathbf{T}(t,0) = \begin{pmatrix}\mathbf{T}_+(0,t) & 0\\ 0 & \mathbf{T}_-(0,t)\end{pmatrix}\begin{pmatrix}\mathbf{x} & 0\\ 0 & \mathbf{x}\end{pmatrix}\begin{pmatrix}\mathbf{T}_+(t,0) & 0\\ 0 & \mathbf{T}_-(t,0)\end{pmatrix}$$
$$= \begin{pmatrix}\mathbf{T}_+(0,t)\mathbf{x}\mathbf{T}_+(t,0) & 0\\ 0 & \mathbf{T}_-(t,0)\mathbf{x}\mathbf{T}_-(t,0)\end{pmatrix} = \begin{pmatrix}\overline{\mathbf{x}}_+ & 0\\ 0 & \overline{\mathbf{x}}_-\end{pmatrix}$$

where:

$$\overline{\mathbf{x}}_+ = \mathbf{T}_+(0,t)\mathbf{x}\mathbf{T}_+(t,0) = \sqrt{\tfrac{\hbar}{2m\omega_+}}\,(\mathbf{a}e^{-i\omega_+t} + \mathbf{a}^\dagger e^{i\omega_+t})$$
$$\overline{\mathbf{x}}_- = \mathbf{T}_-(t,0)\mathbf{x}\mathbf{T}_-(t,0) = \sqrt{\tfrac{\hbar}{2m\omega_-}}\,(\mathbf{b}e^{-i\omega_-t} + \mathbf{b}^\dagger e^{i\omega_-t})$$

The last equalities follow from the relations:

$$\mathbf{T}_+(0,t)\mathbf{a}\mathbf{T}_+(t,0) = \mathbf{a}e^{-i\omega_+t}$$
$$\mathbf{T}_-(0,t)\mathbf{b}\mathbf{T}_-(t,0) = \mathbf{b}e^{-i\omega_-t}$$

and their Hermitian conjugates (see below for more discussion on these realtions).

The ground states of $\mathbf{H}_+$ and $\mathbf{H}_-$ are related via: $|0_+\rangle = \mathbf{S}(\xi)|0_-\rangle$, and earlier I showed that $\mathbf{S}^\dagger(\xi)\mathbf{D}_+(\alpha)\mathbf{S}(\xi) = \mathbf{D}_-(\alpha)$ so we get:

$$\langle\mathbf{x}\rangle_t = \langle\Psi(t)|\mathbf{x}|\Psi(t)\rangle = \langle\Psi(0)|\overline{\mathbf{x}}|\Psi(0)\rangle = \langle\Psi(0)|\overline{\mathbf{x}}|\Psi(0)\rangle$$
$$= \tfrac{1}{2}[\langle 0_+|\mathbf{D}^\dagger_+(\alpha)\overline{\mathbf{x}}_+\mathbf{D}_+(\alpha)|0_+\rangle + \langle 0_-|\mathbf{S}^\dagger(\xi)\mathbf{D}^\dagger_+(\alpha)\overline{\mathbf{x}}_-\mathbf{D}_+(\boldsymbol{\alpha})\mathbf{S}(\xi)|0_-\rangle]$$
$$= \tfrac{1}{2}[\langle 0_+|\mathbf{D}^\dagger_+(\alpha)\overline{\mathbf{x}}_+\mathbf{D}_+(\alpha)|0_+\rangle + \langle 0_-|\mathbf{D}^\dagger_-(\alpha)\mathbf{S}^\dagger(\xi)\overline{\mathbf{x}}_-\mathbf{S}(\xi)\mathbf{D}_-(\boldsymbol{\alpha})|0_-\rangle]$$
$$= \tfrac{1}{2}[\langle\alpha_+|\overline{\mathbf{x}}_+|\alpha_+\rangle + \langle\alpha_-|\mathbf{S}^\dagger(\xi)\overline{\mathbf{x}}_-\mathbf{S}(\xi)|\alpha_-\rangle]$$
$$= \tfrac{1}{2}\left[\langle\alpha_+|\sqrt{\tfrac{\hbar}{2m\omega_+}}\,(\mathbf{a}e^{-i\omega_+t} + \mathbf{a}^\dagger e^{i\omega_+t})|\alpha_+\rangle + \langle\alpha_-|\mathbf{S}^\dagger(\xi)\sqrt{\tfrac{\hbar}{2m\omega_-}}\,(\mathbf{b}e^{-i\omega_-t} + \mathbf{b}^\dagger e^{i\omega_-t})\mathbf{S}(\xi)|\alpha_-\rangle\right]$$

$$= \frac{1}{2}\left[\sqrt{\frac{\hbar}{2m\omega_+}}\langle\alpha_+|(\mathbf{a}e^{-i\omega_+t} + \mathbf{a}^\dagger e^{i\omega_+t})|\alpha_+\rangle + \sqrt{\frac{\hbar}{2m\omega_-}}\langle\alpha_-|\mathbf{S}^\dagger(\xi)(\mathbf{b}e^{-i\omega_-t} + \mathbf{b}^\dagger e^{i\omega_-t})\mathbf{S}(\xi)|\alpha_-\rangle\right]$$

The first half of this is familiar:

$$\langle\alpha_+|(\mathbf{a}e^{-i\omega_+t} + \mathbf{a}^\dagger e^{i\omega_+t})|\alpha_+\rangle = (\langle\alpha_+|\mathbf{a}|\alpha_+\rangle e^{-i\omega_+t} + \langle\alpha_+|\mathbf{a}^\dagger|\alpha_+\rangle e^{i\omega_+t}) = (\alpha e^{-i\omega_+t} + \alpha^* e^{i\omega_+t})\langle\alpha_+|\alpha_+\rangle$$

$$= (\alpha e^{-i\omega_+t} + \alpha^* e^{i\omega_+t}) = 2\,\mathrm{Re}[\alpha e^{-i\omega_+t}]$$

For the second half, we need to do a little more work. The results of Problem 1(a) give:

$$\mathbf{S}^\dagger(\xi)(\mathbf{b}e^{-i\omega_-t} + \mathbf{b}^\dagger e^{i\omega_-t})\mathbf{S}(\xi)$$

$$= (\mathbf{b}\cosh[\xi] - \mathbf{b}^\dagger\sinh[\xi])e^{-i\omega_-t} + (\mathbf{b}^\dagger\cosh[\xi] - \mathbf{b}\sinh[\xi])e^{i\omega_-t}$$

$$= \mathbf{b}(\cosh[\xi]e^{-i\omega_-t} - \sinh[\xi]e^{i\omega_-t}) + \mathbf{b}^\dagger(\cosh[\xi]e^{i\omega_-t} - \sinh[\xi]e^{-i\omega_-t})$$

so letting $\eta = \cosh[\xi]e^{-i\omega_-t} - \sinh[\xi]e^{i\omega_-t}$ we get:

$$\langle\alpha_-|\mathbf{S}^\dagger(\xi)(\mathbf{b}e^{-i\omega_-t} + \mathbf{b}^\dagger e^{i\omega_-t})\mathbf{S}(\xi)|\alpha_-\rangle = \langle\alpha_-|(\eta\mathbf{b} + \eta^*\mathbf{b}^\dagger)|\alpha_-\rangle = (\eta\langle\alpha_-|\mathbf{b}|\alpha_-\rangle + \eta^*\langle\alpha_-|\mathbf{b}^\dagger|\alpha_-\rangle)$$

$$= (\eta\alpha + \eta^*\alpha^*) = 2\,\mathrm{Re}[\eta\alpha]$$

$$= 2\,\mathrm{Re}[\alpha(\cosh[\xi]e^{-i\omega_-t} - \sinh[\xi]e^{i\omega_-t})]$$

Now put it all together and simplify:

$$\langle\mathbf{x}\rangle_t = \left(\sqrt{\frac{\hbar}{2m\omega_+}}\,\mathrm{Re}[\alpha e^{-i\omega_+t}] + \sqrt{\frac{\hbar}{2m\omega_-}}\,\mathrm{Re}[\alpha(\cosh[\xi]e^{-i\omega_-t} - \sinh[\xi]e^{i\omega_-t})]\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega_+}}\,\mathrm{Re}\left[\alpha\left(e^{-i\omega_+t} + \sqrt{\frac{\omega_+}{\omega_-}}(\cosh[\xi]e^{-i\omega_-t} - \sinh[\xi]e^{i\omega_-t})\right)\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega_+}}\begin{pmatrix} \mathrm{Re}[\alpha]\left(\cos(\omega_+t) + \cos(\omega_-t)\sqrt{\frac{\omega_+}{\omega_-}}(\cosh[\xi] - \sinh[\xi])\right) \\ +\mathrm{Im}[\alpha]\left(\sin(\omega_+t) + \sin(\omega_-t)\sqrt{\frac{\omega_+}{\omega_-}}(\cosh[\xi] + \sinh[\xi])\right)\end{pmatrix}$$

$$= \sqrt{\frac{\hbar}{2m\omega_+}}\begin{pmatrix} \mathrm{Re}[\alpha]\left(\cos(\omega_+t) + \left(\sqrt{\frac{\omega_+}{\omega_-}}\exp[-\xi]\right)\cos(\omega_-t)\right) \\ +\mathrm{Im}[\alpha]\left(\sin(\omega_+t) + \left(\sqrt{\frac{\omega_+}{\omega_-}}\exp[\xi]\right)\sin(\omega_-t)\right)\end{pmatrix}$$

$$= \sqrt{\frac{\hbar}{2m\omega_+}}(\mathrm{Re}[\alpha](\cos(\omega_+t) + \cos(\omega_-t)) + \mathrm{Im}[\alpha](\sin(\omega_+t) + \tfrac{\omega_+}{\omega_-}\sin(\omega_-t)))$$

The relations that I asserted above:

$\mathbf{T}_+(0,t)\mathbf{a}\mathbf{T}_+(t,0) = \mathbf{a}e^{-i\omega_+t}$

$\mathbf{T}_-(0,t)\mathbf{b}\mathbf{T}_-(t,0) = \mathbf{b}e^{-i\omega_-t}$

can easily be derived two ways (and are not too hard to memorize either). One way is to write out the series for $\exp[i\omega_+t\mathbf{a}^\dagger\mathbf{a}]$ in the equation $\mathbf{T}_+(0,t)\mathbf{a}\mathbf{T}_+(t,0) = \exp\left[i\frac{t}{\hbar}\mathbf{H}_+\right]\mathbf{a}\exp\left[-i\frac{t}{\hbar}\mathbf{H}_+\right] = \exp[i\omega_+t\mathbf{a}^\dagger\mathbf{a}]\mathbf{a}\exp[-i\omega_+t\mathbf{a}^\dagger\mathbf{a}]$ and explicitly commute **a** through it term by term (and similarly for $\exp\left[i\frac{t}{\hbar}\mathbf{H}_-\right]$ and **b**). Another way is to apply the equation of motion for an operator in the Heisenberg picture; I will do this calculation for a Hamiltonian $\mathbf{H} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$ and $\overline{\mathbf{a}}(t) = \mathbf{T}(0,t)\mathbf{a}\mathbf{T}(t,0)$ with the understanding that it applies to our problem in a straightforward manner by relabeling terms where appropriate:

$$i\hbar\frac{d\overline{\mathbf{a}}(t)}{dt} = \left[\overline{\mathbf{a}}(t), \overline{\mathbf{H}}\right] = \hbar\omega[\overline{\mathbf{a}}(t), \overline{\mathbf{a}}^\dagger(t)\overline{\mathbf{a}}(t)] =$$

$$\hbar\omega\mathbf{T}(0,t)[\mathbf{a}, \mathbf{a}^\dagger\mathbf{a}]\mathbf{T}(t,0) = -\hbar\omega\mathbf{T}(0,t)\mathbf{a}\mathbf{T}(t,0) = -\hbar\omega\overline{\mathbf{a}}(t)$$

So we just solve the differential equation: $\frac{d\overline{\mathbf{a}}(t)}{dt} + i\omega\overline{\mathbf{a}}(t) = 0$ with the initial condition $\overline{\mathbf{a}}(0) = \mathbf{a}$ to get: $\overline{\mathbf{a}}(t) = \mathbf{a}e^{-i\omega t}$.

If you did not immediately transform to the Heisenberg picture, you could still use these relations when you get to the point:

$$\langle \mathbf{x} \rangle_t = \tfrac{1}{2}[\langle 0_+|\mathbf{D}_+^\dagger(\alpha)\mathbf{T}_+(0,t)\mathbf{x}\mathbf{T}_+(t,0)\mathbf{D}_+(\alpha)|0_+\rangle + \langle 0_-|\mathbf{S}^\dagger(\xi)\mathbf{D}_+^\dagger(\alpha)\mathbf{T}_-(0,t)\mathbf{x}\mathbf{T}_-(t,0)\mathbf{D}_+(\boldsymbol{\alpha})\mathbf{S}(\xi)|0_-\rangle]$$

and it would be essentially be the same thing. (I originally was going to do it this way, but decided it would be more clear that I was in fact using the Heisenberg picture if I switched immediately.)

If you were not familiar with these relations, then you probably did not even consider switching to the Heisenberg picture, and you had to figure out (or hopefully remember from class) how $\mathbf{T}_-(t,0)$ commutes through $\mathbf{S}(\xi)$. I will show this method as well:

$$|\Psi(t)\rangle = \mathbf{T}(t,0)|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{T}_+(t,0) & 0 \\ 0 & \mathbf{T}_-(t,0) \end{pmatrix}\begin{pmatrix} \mathbf{D}_+(\alpha)|0_+\rangle \\ \mathbf{D}_+(\alpha)|0_+\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{T}_+(t,0)\mathbf{D}_+(\alpha)|0_+\rangle \\ \mathbf{T}_-(t,0)\mathbf{D}_+(\alpha)|0_+\rangle \end{pmatrix}$$

$$\begin{aligned}\langle \mathbf{x} \rangle_t &= \langle \Psi(t)|\mathbf{x}|\Psi(t)\rangle \\ &= \tfrac{1}{2}[\langle 0_+|\mathbf{D}_+^\dagger(\alpha)\mathbf{T}_+(0,t)\mathbf{x}\mathbf{T}_+(\mathbf{t},\mathbf{0})\mathbf{D}_+(\alpha)|0_+\rangle + \langle 0_+|\mathbf{D}_+^\dagger(\alpha)\mathbf{T}_-(0,t)\mathbf{x}\mathbf{T}_-(\mathbf{t},\mathbf{0})\mathbf{D}_+(\alpha)|0_+\rangle] \\ &= \tfrac{1}{2}[\langle 0_+|\mathbf{D}_+^\dagger(\alpha)\mathbf{T}_+(0,t)\mathbf{x}\mathbf{T}_+(\mathbf{t},\mathbf{0})\mathbf{D}_+(\alpha)|0_+\rangle + \langle 0_-|\mathbf{S}^\dagger(\xi)\mathbf{D}_+^\dagger(\alpha)\mathbf{T}_-(0,t)\mathbf{x}\mathbf{T}_-(\mathbf{t},\mathbf{0})\mathbf{D}_+(\alpha)\mathbf{S}(\xi)|0_-\rangle]\end{aligned}$$

The first half of this is familiar:

$$\begin{aligned}\langle 0_+|\mathbf{D}_+^\dagger(\alpha)\mathbf{T}_+(0,t)\mathbf{x}\mathbf{T}_+(\mathbf{t},\mathbf{0})\mathbf{D}_+(\alpha)|0_+\rangle &= \langle \alpha e^{-i\omega_+ t}|\sqrt{\tfrac{\hbar}{2m\omega_+}}(\mathbf{a}+\mathbf{a}^\dagger)|\alpha e^{-i\omega_+ t}\rangle \\ &= \sqrt{\tfrac{\hbar}{2m\omega_+}}(\alpha e^{-i\omega_+ t} + \alpha^* e^{i\omega_+ t})\langle \alpha e^{-i\omega_+ t}|\alpha e^{-i\omega_+ t}\rangle = \sqrt{\tfrac{2\hbar}{m\omega_+}}\,\mathrm{Re}[\alpha e^{-i\omega_+ t}]\end{aligned}$$

For the second half, use:

$$\mathbf{S}^\dagger(\xi)\mathbf{D}_+(\alpha)\mathbf{S}(\xi) = \mathbf{D}_-(\alpha)$$

and in class we found:

$$\mathbf{T}_-(t,0)\mathbf{S}(\xi) = \mathbf{S}(\xi e^{-i2\omega_- t})\mathbf{T}_-(t,0)$$

this is correct up to a phase (I think), but the phase will cancel with the phase of the conjugate expression, so I won't bother to figure it out. Then we get:

$$\begin{aligned}&\langle 0_-|\mathbf{S}^\dagger(\xi)\mathbf{D}_+^\dagger(\alpha)\mathbf{T}_-(0,t)\mathbf{x}\mathbf{T}_-(t,0)\mathbf{D}_+(\alpha)\mathbf{S}(\xi)|0_-\rangle \\ &= \langle 0_-|\mathbf{D}_-^\dagger(\alpha)\mathbf{T}_-(0,t)\mathbf{S}^\dagger(\xi e^{-i2\omega_- t})\mathbf{x}\mathbf{S}(\xi e^{-i2\omega_- t})\mathbf{T}_-(t,0)\mathbf{D}_-(\alpha)|0_-\rangle \\ &= \langle \alpha e^{-i\omega_- t}|\mathbf{S}^\dagger(\xi e^{-i2\omega_- t})\mathbf{x}\mathbf{S}(\xi e^{-i2\omega_- t})|\alpha e^{-i\omega_- t}\rangle \\ &= \sqrt{\tfrac{\hbar}{2m\omega_-}}\langle \alpha e^{-i\omega_- t}|\mathbf{S}^\dagger(\xi e^{-i2\omega_- t})(\mathbf{b}+\mathbf{b}^\dagger)\mathbf{S}(\xi e^{-i2\omega_- t})|\alpha e^{-i\omega_- t}\rangle \\ &= \sqrt{\tfrac{\hbar}{2m\omega_-}}\langle \alpha e^{-i\omega_- t}|(\mathbf{b}\cosh[\xi] - \mathbf{b}^\dagger\exp[-i2\omega_- t]\sinh[\xi] + \mathbf{b}^\dagger\cosh[\xi] - \mathbf{b}\exp[i2\omega_- t]\sinh[\xi])|\alpha e^{-i\omega_- t} \\ &= \sqrt{\tfrac{\hbar}{2m\omega_-}}[(\alpha e^{-i\omega_- t} + \alpha^* e^{i\omega_- t})\cosh[\xi] - (\alpha e^{i\omega_- t} + \alpha^* e^{-i\omega_- t})\sinh[\xi]] \\ &= \sqrt{\tfrac{\hbar}{2m\omega_-}}\,\mathrm{Re}[\alpha(\cosh[\xi]e^{-i\omega_- t} - \sinh[\xi]e^{i\omega_- t})]\end{aligned}$$

These are both exactly the same results obtained before, so the answer is the same:

$$\langle \mathbf{x} \rangle_t = \sqrt{\tfrac{\hbar}{2m\omega_+}}(\mathrm{Re}[\alpha](\cos(\omega_+ t) + \cos(\omega_- t)) + \mathrm{Im}[\alpha](\sin(\omega_+ t) + \tfrac{\omega_+}{\omega_-}\sin(\omega_- t)))$$

Finally, here is the finesse route:

As before, we know:

$$\langle \mathbf{x} \rangle_t = \tfrac{1}{2}(\langle \mathbf{x}_+ \rangle_t + \langle \mathbf{x}_- \rangle_t)$$

$$\langle \mathbf{x}_+ \rangle_t = \langle 0_+|\mathbf{D}_+^\dagger(\alpha)\mathbf{T}_+(0,t)\mathbf{x}\mathbf{T}_+(t,0)\mathbf{D}_+(\alpha)|0_+\rangle$$

$$\langle \mathbf{x}_- \rangle_t = \langle 0_+|\mathbf{D}_+^\dagger(\alpha)\mathbf{T}_-(0,t)\mathbf{x}\mathbf{T}_-(t,0)\mathbf{D}_+(\alpha)|0_+\rangle$$

The time evolution for position expectation value in a LHO gives:

$\langle \mathbf{x}_\pm \rangle_t = \langle \mathbf{x}_\pm \rangle_0 \cos(\omega_\pm t) + \frac{\langle \mathbf{p}_\pm \rangle_0}{m\omega_\pm} \sin(\omega_\pm t)$

Then recall the expectation values for a coherent state:

$\langle \mathbf{x} \rangle = \langle \alpha | \mathbf{x} | \alpha \rangle = \sqrt{\frac{2\hbar}{m\omega}} \, \mathrm{Re}[\alpha]$

$\langle \mathbf{p} \rangle = \langle \alpha | \mathbf{p} | \alpha \rangle = \sqrt{2\hbar m\omega} \, \mathrm{Im}[\alpha]$

which give (for both the + and – initial states):

$\langle \mathbf{x}_\pm \rangle_0 = \langle 0_+ | \mathbf{D}_+^\dagger(\alpha) \mathbf{x} \mathbf{D}_+(\alpha) | 0_+ \rangle = \langle \alpha_+ | \mathbf{x} | \alpha_+ \rangle = \sqrt{\frac{2\hbar}{m\omega_+}} \, \mathrm{Re}[\alpha]$

$\langle \mathbf{p}_\pm \rangle_0 = \langle 0_+ | \mathbf{D}_+^\dagger(\alpha) \mathbf{p} \mathbf{D}_+(\alpha) | 0_+ \rangle = \langle \alpha_+ | \mathbf{p} | \alpha_+ \rangle = \sqrt{2\hbar m\omega_+} \, \mathrm{Im}[\alpha]$

Put it all together:

$$\langle \mathbf{x} \rangle_t = \frac{1}{2} \left(\sqrt{\frac{2\hbar}{m\omega_+}} \, \mathrm{Re}[\alpha](\cos(\omega_+ t) + \cos(\omega_- t)) + \sqrt{2\hbar m\omega_+} \, \mathrm{Im}[\alpha]\left(\frac{1}{m\omega_+} \sin(\omega_+ t) + \frac{1}{m\omega_-} \sin(\omega_- t)\right) \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega_+}} \left(\mathrm{Re}[\alpha](\cos(\omega_+ t) + \cos(\omega_- t)) + \mathrm{Im}[\alpha]\left(\sin(\omega_+ t) + \frac{\omega_+}{\omega_-} \sin(\omega_- t)\right)\right)$$