Ph 195c Midterm Solutions:

Problem 1

(a) For the delta-shell potential, the wavefunction in the $r \neq a$ regions is simply the solution to the free Schrödinger equation and can be written as Bessel function/Legendre polynomial expansions (as with all central potential scattering, we only consider the $m = 0$ terms of the spherical harmonics). The interior ($r < a$) wavefunction can be expanded in terms of spherical Bessel functions (the spherical Neumann functions can be dropped since they diverge at $r = 0$):

$$\Psi_{int} = \sum_{l=0}^{\infty} B_l(2l+1) j_l(kr) P_l(\cos\theta)$$

The expansion of the external ($r > a$) wavefunction is given by:

$$\begin{aligned}\Psi_{ext} &= N\left[\left(\sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)\right) + \left(\sum_{l=0}^{\infty} C_l(2l+1)h_l^{(1)}(kr)P_l(\cos\theta)\right)\right] \\ &= N\left[\sum_{l=0}^{\infty}\left((2l+1)i^l j_l(kr) + C_l(2l+1)h_l^{(1)}(kr)\right)P_l(\cos\theta)\right]\end{aligned}$$

The first line of the exterior expansion shows that this has the correct asymptotic form for scattering. In particular:

$$e^{ikz} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$

and the asymptotic behavior $h_l^{(1)}(kr) \sim (-i)^{l+1}\frac{e^{ikr}}{kr}$ gives the second summation's behavior:

$$\sum_{l=0}^{\infty} C_l(2l+1)h_l^{(1)}(kr)P_l(\cos\theta) \sim \left(\frac{1}{k}\sum_{l=0}^{\infty} C_l(-i)^{l+1}(2l+1)P_l(\cos\theta)\right)\frac{e^{ikr}}{r}$$

which, when compared to the scattering asymptotic form:

$$\Psi \sim N\left[e^{ikz} + f_k(\theta)\frac{e^{ikr}}{r}\right]$$

$$f_k(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin(\delta_l)P_l(\cos\theta)$$

gives the relation:

$$|C_l|^2 = \sin^2\delta_l$$

To find the solution, we must satisfy the boundary conditions term by term. However, we are only interested in the s-wave, so we only need to consider the $l = 0$ term.

$$\Psi_{int}(r = a) = \Psi_{ext}(r = a) \Rightarrow N\left[j_0(ka) + C_0 h_0^{(1)}(ka)\right] = B_0 j_0(ka)$$

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{a-\epsilon}^{a+\epsilon}\left\{\frac{d}{dr}\left(r^2\frac{d}{dr}\Psi\right) - l(l+1)\Psi - \frac{2mr^2}{\hbar^2}(V(r)-E)\Psi = 0\right\}dr \\ &\quad\Rightarrow a^2\left(Nk\left[j_0'(ka) + C_0 h_0^{(1)\prime}(ka)\right] - B_0 k j_0'(ka)\right) - \frac{2ma^2}{\hbar^2}g(B_0 j_0(ka)) = 0 \\ &\quad\Rightarrow N\left[j_0'(ka) + C_0 h_0^{(1)\prime}(ka)\right] - B_0\left(j_0'(ka) + \frac{2mg}{\hbar^2 k}j_0(ka)\right) = 0\end{aligned}$$

Notice if $j_0(ka) = 0$ (i.e. if $ka = n\pi$), then we immediately get: $C_0 = 0$ and $B_0 = N$, which gives a (s-wave) wavefunction: $\Psi_{l=0}(\vec{r}) = Nj_0(kr)$ which has no scattering ($f_{k,l=0}(\theta) = 0, \sigma_0 = 0$).

Now assuming $j_0(ka) \neq 0$, we get:

$$B_0 = N\left[j_0(ka) + C_0 h_0^{(1)}(ka)\right]\frac{1}{j_0(ka)}$$

$$j_0'(ka) + C_0 h_0^{(1)\prime}(ka) - \left(j_0(ka) + C_0 h_0^{(1)}(ka)\right)\left(\frac{j_0'(ka)}{j_0(ka)} + \frac{2mg}{\hbar^2 k}\right) = 0$$

$$C_0 = \frac{2mg}{\hbar^2 k} j_0(ka)\left[h_0^{(1)\prime}(ka) - \left(\frac{j_0'(ka)}{j_0(ka)} + \frac{2mg}{\hbar^2 k}\right)h_0^{(1)}(ka)\right]^{-1}$$

Now using the definitions of the $l = 0$ spherical Bessel and Hankel functions: $j_0(x) = \frac{\sin x}{x}$ and $h_0^{(1)}(x) = -i\frac{e^{ix}}{x}$ we get:

$$C_0 = \frac{2mg}{\hbar^2 k}\sin(ka)\left[e^{ika} + i\frac{e^{ika}}{ka} + i\left(\cot(ka) - \frac{1}{ka} + \frac{2mg}{\hbar^2 k}\right)e^{ika}\right]^{-1}$$

$$= \frac{2mg}{\hbar^2 k}\sin(ka)e^{-ika}\left[1 + i\left(\cot(ka) + \frac{2mg}{\hbar^2 k}\right)\right]^{-1}$$

Now we can plug this into the equation for the total scattering cross section (in terms of partial waves):

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}|C_0|^2 = \frac{4\pi}{k^2}\left(\frac{2mg}{\hbar^2 k}\right)^2 \frac{\sin^2(ka)}{1+\left(\cot(ka)+\frac{2mg}{\hbar^2 k}\right)^2} = \frac{16\pi m^2 g^2}{\hbar^4 k^4}\frac{\sin^4(ka)}{1+\frac{2mg}{\hbar^2 k}\sin(2ka)+\frac{4m^2g^2}{\hbar^4k^2}\sin^2(ka)}$$

$$= \frac{4\pi g^2}{E^2}\frac{\sin^4\left(\sqrt{\frac{2ma^2}{\hbar^2}}E^{1/2}\right)}{1+\sqrt{\frac{2mg^2}{\hbar^2}}E^{-1/2}\sin\left(2\sqrt{\frac{2ma^2}{\hbar^2}}E^{1/2}\right)+\frac{2mg^2}{\hbar^2}E^{-1}\sin^2\left(\sqrt{\frac{2ma^2}{\hbar^2}}E^{1/2}\right)}$$

Notice that σ_0 is finite as $E \to 0$ ($k \to 0$) and zero when $E \to \frac{\pi^2\hbar^2 n^2}{2ma^2}$ (i.e. $ka \to n\pi$) so it is well-behaved for all positive energies (since these are the only eneries where the behavior is in question).

There are several different ways you could have solved this problem (for example: using the u_{kl} radial equation rather than the R_{kl} radial equation, using $j_0(kr)$ and $n_0(kr)$ ($\sin(kr)$ and $\cos(kr)$) or $h_0^{(1)}(kr)$ and $h_0^{(2)}(kr)$ (e^{ikr} and e^{-ikr}) rather than my $j_0(kr)$ and $h_0^{(1)}(kr)$ hybrid, etc.), but the key point is to make sure that you identify the correct relation between the coefficients in your solution (after imposing boundary conditions) and the scattering phase shift δ_0. Other possible correct forms (with respect to the value of C_0 that I found above) are:

$$e^{i2\delta_0} = 1 + 2C_0$$

$$\tan\delta_0 = \frac{-iC_0}{1+C_0} = -\left(\frac{2mg}{\hbar^2 k}\right)\frac{\sin^2(ka)}{1+\frac{2mg}{\hbar^2 k}\sin(ka)\cos(ka)}$$

$$\delta_0 = \arctan\left(\frac{1}{\cot(ka)+\frac{2mg}{\hbar^2 k}}\right) - ka$$

$$\sin^2\delta_0 = -\frac{C_0^2}{1+2C_0}$$

(b) The scattering resonances are found where the cross-section peaks, attaining its maximum value of $\sigma_l = \frac{4\pi}{k^2}(2l+1)$ (i.e. $\delta_l = n\pi + \frac{\pi}{2}$). Using one of the relations above, we see that for the s-wave this corresponds to $C_0 = -1$, which can be further reduced to the equation: $\sin(2ka) = -\frac{\hbar^2}{mg}k$. From this we can see that there is a minimum g below which there are no resonances given by: $g_{\min} = \frac{\hbar^2}{2ma}\sqrt{1 + \left(\frac{3\pi}{2}\right)^2}$. Additionally, since g is finite, there will be a finite number of resonances, and the larger the value of g the more resonances there will be.

As mentioned in part (a), there is a sort of anti-resonance behavior when $E = \frac{\pi^2\hbar^2 n^2}{2ma^2}$ which are the energies that have zero s-wave scattering (i.e. $ka = n\pi$).

If g were negative, you would find poles of the scattering amplitude (in the negative energy range) determined by the equation: $\coth(\kappa a) = \pm 1 - \left(\frac{2mg}{\hbar^2}\right)\frac{1}{\kappa a}$ where $\kappa = \sqrt{\frac{-2mE}{\hbar^2}}$ is positive. (Notice this equation shows that there are no poles for the $g > 0$ case.)

(c) The contribution from $l > 0$ partial waves is negligible when $ka \ll \sqrt{l(l+1)} \sim 1$, so the energy condition becomes: $E \ll \frac{\hbar^2}{ma^2}$

Problem 2

(a) I will denote states in the Hilbert space as $|\Psi\rangle = |s\rangle \otimes |\psi\rangle = |s;\psi\rangle$. The first thing to do is examine the potential: $\mathbf{V}_1 = \frac{1}{2}m\omega^2|-\rangle\langle-| \otimes \left|\vec{\mathbf{r}}\right|^2$, where I am using $\frac{1}{2}m\omega^2$ rather than $\frac{1}{2}k$ in order to save the symbol k for representing the wavenumber. This potential corresponds to states of positive spin $|+\rangle$ being in a free potential and states of negative spin $|-\rangle$ being in a harmonic oscillator potential. The wavefunctions for the 3-D harmonic oscillator can be obtained by separation of variables in cartesian coordinates, giving: $|n_x, n_y, n_z\rangle = |n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle$ (i.e. $\psi_{n_x,n_y,n_z}(\vec{x}) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z)$) with energies $E_{n_x,n_y,n_z} = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)$ for $n_x, n_y, n_z = 0, 1, 2, \ldots$ (you should be familiar with this). Alternatively, you could obtain the wavefunction in spherical coordinates giving the eigenfunctions: $\psi_{n,l,m}(\vec{x}) = R_{n,l}(r)Y_{l,m}(\theta,\varphi)$ where $R_{n,l}(r) = r^l f_{n,l}(r)e^{-\beta r^2}$ with $\beta = \frac{m\omega}{2\hbar}$ and $f_{n,l}(r) = \sum_k a_k r^k$ is defined by the recursion relation: $a_{k+2} = 2\beta\frac{2k-4n}{(k+2)(k+2l+3)}a_k$ for $k \geq 0$ even, a_0 is determined by normalization, and $a_k = 0$ for all other k and the energies are $E_{n,l,m} = \hbar\omega\left(2n + l + \frac{3}{2}\right)$ for $n, l = 0, 1, 2\ldots$ and $m = -l, -l+1, \ldots, l-1, l$. (I expect most of you are not familiar with this form, but you don't have to be to solve this problem. It is interesting to note that this gives another explanation for the energy level degeneracies of the 3-D HO, $d(N) = \frac{(N+2)(N+1)}{2}$ for $E = \hbar\omega\left(N + \frac{3}{2}\right)$.) My reason for mentioning the spherical coordinate eigenfunctions was to specifically make clear that the ground state:

$$\left\langle \vec{x}|0\right\rangle = \psi_{n_x=0,n_y=0,n_z=0}(\vec{x}) = \psi_{n=0,l=0,m=0}(\vec{x}) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-\frac{m\omega}{2\hbar}(x^2+y^2+z^2)} = \left(\frac{2\beta}{\pi}\right)^{3/4} e^{-\beta r^2}$$

is a $l = m = 0$ state (but you could also deduce this from the fact that it has no angular dependence when written in spherical coordinates).

In order to better demonstrate how to find transition rates (and to double check my results) I will calculate the transition rate using two different options: transitions to plane waves and transitions to spherical waves. I will define the continuum states $\left|\vec{k}\right\rangle$ and $|k\rangle$ to be the plane wave and spherical s-wave (respectively) with delta function normalizations:

$$\left\langle \vec{x}|\vec{k}\right\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\vec{k}\cdot\vec{x}}$$

$$\left\langle \vec{k}|\vec{k}'\right\rangle = \frac{1}{(2\pi)^3}\int d^3\vec{x}\left[e^{i\left(\vec{k}'-\vec{k}\right)\cdot\vec{x}}\right] = \delta^3\left(\vec{k} - \vec{k}'\right)$$

$$\left\langle \vec{x}|k\right\rangle = \frac{k}{\pi\sqrt{2}}j_o(kr) = \frac{1}{\pi\sqrt{2}}\frac{\sin(kr)}{r}$$

$$\langle k|k'\rangle = \frac{kk'}{2\pi^2}\int d^3\vec{x}[j_o(kr)j_o(k'r)] = \frac{2}{\pi}\int_0^\infty dr[\sin(kr)\sin(k'r)] = \delta(k - k')$$

We only need to consider the spherical s-wave, because the 3-D HO ground state is a $l = m = 0$ state and so has vanishing inner product with spherical harmonic states that do not have $l = m = 0$. Taking inner products of these continuum states with the HO ground state, we have (with $\left|\vec{k}\right| = k$ for the plane wave case):

$$\left\langle 0|\vec{k}\right\rangle = \int d^3\vec{x}\left[\frac{1}{(2\pi)^{3/2}}e^{i\vec{k}\cdot\vec{x}}\left(\frac{2\beta}{\pi}\right)^{3/4}e^{-\beta|\vec{x}|^2}\right] = \left(\frac{\beta}{2\pi^3}\right)^{3/4}\int_0^\infty dr\int_0^\pi d\theta\int_0^{2\pi}d\varphi\left[r^2\sin\theta e^{ikr\cos\theta}e^{-\beta r^2}\right]$$
$$= \left(\frac{\beta}{2\pi^3}\right)^{3/4}2\pi\int_0^\infty dr\left[r^2e^{-\beta r^2}\frac{e^{ikr\cos\theta}}{-ikr}\Big|_0^\pi\right] = \left(\frac{\beta}{2\pi^3}\right)^{3/4}\frac{4\pi}{k}\int_0^\infty dr\left[r\sin(kr)e^{-\beta r^2}\right] = \left(\frac{\beta}{2\pi^3}\right)^{3/4}\frac{4\pi}{k}I$$
$$= \left(\frac{1}{2\pi\beta}\right)^{3/4}\exp\left[-\frac{k^2}{4\beta}\right]$$

$$\langle 0|k\rangle = \int d^3\vec{x}\left[\frac{k}{\pi\sqrt{2}}j_o(kr)\left(\frac{2\beta}{\pi}\right)^{3/4}e^{-\beta r^2}\right] = \left(\frac{2\beta}{\pi}\right)^{3/4}\frac{1}{\pi\sqrt{2}}\int_0^\infty dr\int_0^\pi d\theta\int_0^{2\pi}d\varphi\left[r^2\sin\theta\frac{\sin(kr)}{r}e^{-\beta r^2}\right]$$
$$= \left(\frac{2\beta}{\pi}\right)^{3/4}2\sqrt{2}\int_0^\infty dr\left[r\sin(kr)e^{-\beta r^2}\right] = \left(\frac{2\beta}{\pi}\right)^{3/4}2\sqrt{2}I$$
$$= \left(\frac{2}{\pi\beta}\right)^{3/4}\frac{k\sqrt{2\pi}}{2}\exp\left[-\frac{k^2}{4\beta}\right]$$

Where I used the following evaluation of the integral:

$$I = \int_0^\infty dr\left[r\sin(kr)e^{-\beta r^2}\right] = -\frac{1}{2\beta}\sin(kr)e^{-\beta r^2}\Big|_0^\infty + \frac{k}{2\beta}\int_0^\infty dr\left[\cos(kr)e^{-\beta r^2}\right]$$
$$= \frac{k}{2\beta}\int_0^\infty dr\left[\cos(kr)e^{-\beta r^2}\right] = \frac{k}{2\beta}\int_0^\infty dr\left[\frac{1}{2}(e^{ikr}+e^{-ikr})e^{-\beta r^2}\right]$$
$$= \frac{k}{4\beta}\left(\int_0^\infty dr\left[e^{ikr}e^{-\beta r^2}\right]+\int_{-\infty}^0 dr\left[e^{ikr}e^{-\beta r^2}\right]\right) = \frac{k}{4\beta}\int_{-\infty}^\infty dr\left[e^{ikr}e^{-\beta r^2}\right]$$
$$= \frac{k}{4\beta}\exp\left[-\frac{k^2}{4\beta}\right]\int_{-\infty}^\infty dr\left[e^{-\beta\left(r-i\frac{k}{2\beta}\right)^2}\right] = \frac{k\sqrt{\pi}}{4\beta^{3/2}}\exp\left[-\frac{k^2}{4\beta}\right]$$

Now it remains to find the density of states for $\left|\vec{k}\right\rangle$ and $|k\rangle$.

For $\left|\vec{k}\right\rangle$, we use $\rho_f\left(\hat{k},E\right)dEd\hat{k} = d^3\vec{k}$ where $\hat{k} = \frac{\vec{k}}{\left|\vec{k}\right|}$ is the direction of the wavevector and $d\hat{k}$ corresponds to the differential solid angle (I did not want to use Ω and $d\Omega$ since the symbol Ω was already used in **W**) and $E = \frac{\hbar^2k^2}{2m}$ to get:

$$\rho_f\left(\hat{k},E\right) = \frac{k^2dkd\hat{k}}{dEd\hat{k}} = \frac{k^2dk}{\frac{\hbar^2}{m}kdk} = \frac{mk}{\hbar^2} = \frac{\sqrt{2}m^{3/2}}{\hbar^3}E^{1/2}$$

Finally, we plug everything into Fermi's Golden Rule with $|\Psi_0\rangle = |-\rangle\otimes|0\rangle$, $\left|\Psi_{\vec{k}}\right\rangle = |+\rangle\otimes\left|\vec{k}\right\rangle$, $\vec{k}_0 = k_0\hat{k}$, and $E_0 = \frac{\hbar^2k_0^2}{2m} = \frac{3}{2}\hbar\omega$ to get the transition rate:

$$w\left(0\to\vec{k}_0\right) = \frac{2\pi}{\hbar}\left|\left\langle\Psi_{\vec{k}_0}\right|\mathbf{W}|\Psi_0\rangle\right|^2\rho_f\left(\hat{k},E_0\right) = \frac{2\pi}{\hbar}(\hbar\Omega)^2\left|\left\langle 0|\vec{k}_0\right\rangle\right|^2\rho_f\left(\hat{k},E_0\right)$$
$$= 2\pi\hbar\Omega^2\left(\frac{1}{2\pi\beta}\right)^{3/2}\exp\left[-\frac{k_0^2}{2\beta}\right]\frac{\sqrt{2}m^{3/2}}{\hbar^3}E_0^{1/2} = 2\pi\hbar\Omega^2\left(\frac{\hbar}{\pi m\omega}\right)^{3/2}e^{-3}\frac{\sqrt{2}m^{3/2}}{\hbar^3}\left(\frac{3}{2}\hbar\omega\right)^{1/2}$$
$$= \frac{2\sqrt{3\pi}}{\pi e^3}\frac{\Omega^2}{\omega}$$

Since we are not looking for transitions to plane waves travelling in a particular direction, we integrate over the solid angle of wavevector directions, giving:

$$w(0\to k_0) = \frac{8\sqrt{3\pi}}{e^3}\frac{\Omega^2}{\omega}$$

Now for the spherical s-wave $|k\rangle$, we use $\rho_f(E)dE = dk$ and $E = \frac{\hbar^2k^2}{2m}$ to get:

$$\rho_f(E) = \frac{dk}{dE} = \frac{dk}{\frac{\hbar^2}{m}kdk} = \frac{m}{\hbar^2k} = \frac{\sqrt{m}}{\hbar\sqrt{2}}E^{-1/2}$$

Plugging into Fermi's Golden Rule with $|\Psi_0\rangle = |-\rangle\otimes|0\rangle$, $|\Psi_k\rangle = |+\rangle\otimes|k\rangle$, and $E_0 = \frac{\hbar^2k_0^2}{2m} = \frac{3}{2}\hbar\omega$ gives:

$$w(0\to k_0) = \frac{2\pi}{\hbar}|\langle\Psi_{k_0}|\mathbf{W}|\Psi_0\rangle|^2\rho_f(E_0) = \frac{2\pi}{\hbar}(\hbar\Omega)^2|\langle 0|k_0\rangle|^2\rho_f(E_0)$$
$$= 2\pi\hbar\Omega^2\left(\frac{2}{\pi\beta}\right)^{3/2}\frac{k_0^2\pi}{2}\exp\left[-\frac{k_0^2}{2\beta}\right]\frac{\sqrt{m}}{\hbar\sqrt{2}}E_0^{-1/2} = 2\pi\hbar\Omega^2\left(\frac{4\hbar}{\pi m\omega}\right)^{3/2}\frac{3m\omega\pi}{2\hbar}e^{-3}\frac{\sqrt{m}}{\hbar\sqrt{2}}\left(\frac{3}{2}\hbar\omega\right)^{-1/2}$$
$$= \frac{8\sqrt{3\pi}}{e^3}\frac{\Omega^2}{\omega}$$

Thus, we get the same result whether we use transitions to plane waves or transitions to spherical waves (as expected).

(b) First consider the evolution of a general gaussian in free space and in a HO system. In free space, a gaussian wavepacket simply spreads out, dissipating with time. In the HO potential, a gaussian obeys a periodic evolution (think squeezed states). The frequency of transitions between the positive spin (free potential) and negative spin (HO potential) states are determined by the value of Ω.

In the $\Omega \gg \omega$ limit, there is a rapid transition between the positive and negative spin states. Consequently, the position space wavefunction part of the initial state $|\Psi_0\rangle = |-\rangle \otimes |0\rangle$ remains essentially unchanged for a while, since the spreading effect of being in the positive spin free potential occurs at a rate much slower than the rate at which **W** flips the spin back from positive to negative. Of course after a long time, the initial state will dissipate, which is the unavoidable effect of coupling to the free potential continuum.

In the $\Omega \ll \omega$ limit, there is a slow transition between the positive and negative states, so the rate at which the gaussian wavepacket spreads out is much higher than the rate at which the spin would flip back from positive to negative. When some of the amplitude of the initial state $|\Psi_0\rangle = |-\rangle \otimes |0\rangle$ leaks into the positive spin free potential, it almost completely dissipates since you would expect almost no HO ground state left by the time it flips back to negative spin. Effectively, the decay rate of the initial state is the same as the transition rate w that we solved for in part (a).

(c) For states $|s\rangle \otimes |\psi\rangle = |s;\psi\rangle$ in the Hilbert space, the potential

$$V_2^+(\vec{x}) = \langle +;\vec{x}|\mathbf{V}_2|+;\vec{x}\rangle = \begin{cases} +\infty & \text{for } |\vec{x}| < a \\ 0 & \text{for } |\vec{x}| \geq a \end{cases}$$

$$V_2^-(\vec{x}) = \langle -;\vec{x}|\mathbf{V}_2|-;\vec{x}\rangle = \begin{cases} 0 & \text{for } |\vec{x}| < a \\ +\infty & \text{for } |\vec{x}| \geq a \end{cases}$$

(and spin off-diagonal terms of $\mathbf{V}_2$ being zero) requires that:

$\psi(\vec{x}) = \langle \vec{x}|\psi\rangle = 0$ in the region $|\vec{x}| \leq a$ for states with $s = +$

$\psi(\vec{x}) = \langle \vec{x}|\psi\rangle = 0$ in the region $|\vec{x}| \geq a$ for states with $s = -$

Hence, for any two allowed states with different spin, $|+;\psi\rangle$ and $|-;\varphi\rangle$ (with $\psi(\vec{x}) = \langle \vec{x}|\psi\rangle$ vanishing for $|\vec{x}| \leq a$ and $\varphi(\vec{x}) = \langle \vec{x}|\varphi\rangle$ vanishing for $|\vec{x}| \geq a$), we will have $\langle\psi|\varphi\rangle = \int \psi^*(\vec{x})\varphi(\vec{x})\,d^3\vec{x} = 0$ since no overlap is possible. Together with the fact that $\langle s_1|\boldsymbol{\sigma}_x|s_2\rangle = 1 - \delta_{s_1,s_2}$ this implies:

$$\langle +;\psi|\mathbf{W}|+;\varphi\rangle = \hbar\Omega\langle +|\boldsymbol{\sigma}_x|+\rangle \int \psi^*(\vec{x})\varphi(\vec{x})\,d^3\vec{x} = 0$$
$$\langle +;\psi|\mathbf{W}|-;\varphi\rangle = \hbar\Omega\langle +|\boldsymbol{\sigma}_x|-\rangle \int \psi^*(\vec{x})\varphi(\vec{x})\,d^3\vec{x} = 0$$
$$\langle -;\psi|\mathbf{W}|+;\varphi\rangle = \hbar\Omega\langle -|\boldsymbol{\sigma}_x|+\rangle \int \psi^*(\vec{x})\varphi(\vec{x})\,d^3\vec{x} = 0$$
$$\langle -;\psi|\mathbf{W}|-;\varphi\rangle = \hbar\Omega\langle -|\boldsymbol{\sigma}_x|-\rangle \int \psi^*(\vec{x})\varphi(\vec{x})\,d^3\vec{x} = 0$$

for any allowed $\psi(\vec{x})$ and $\varphi(\vec{x})$, and so it follows that the transition rate vanishes, since **W** is effectively zero when the system has potential $\mathbf{V}_2$.

Problem 3

The principle quantum number n:

Given by $n = N + l + 1$ where $N, l = 0, 1, \ldots$ where N is the degree of the $w(\kappa r)$ polynomial part of the radial solution $R_{n,l}(r) = (\kappa r)^l e^{\kappa r} w(\kappa r)$ and l is the orbital angular momentum quantum number of the angular (spherical harmonic) solution $Y_{l,m}(\theta, \varphi)$ for the electron wavefunction in the hydrogen atom's Coulomb potential. The principle quantum number can take the values $n = 1, 2, \ldots$ and determines the bound state energies: $E_n = -\frac{me^4}{2\hbar^2 n^2}$.
States with quantum number n have possible orbital angular momentum quantum numbers $l = 0, \ldots, n-1$ and so gives rise to the n state degeneracy $d(n) = n^2$.

The five significant angular momenta:

L is the orbital angular momentum of the electron's wavefunction in the atom. It has eigenstates given by the spherical harmonics, taking the quantum numbers (eigenvalues) $l = 0, 1, \ldots$

S is the electron spin (intrinsic) angular momentum. There is no meaningful interpretation of a physical origin of spin in non-relativistic quantum mechanics, and so it is simply considered an intrinsic form of angular momentum. The electron has spin quantum number $s = \frac{1}{2}$. Electron spin can be distiguished from other (e.g. nuclear) spins by comparing magnetic moments. The magnetic moment for electron spin is given by:
$\mathbf{M}_S = -\frac{e}{m_e}\mathbf{S} = \frac{g_e \mu_B}{\hbar}\mathbf{S}, \mu_B = -\frac{e\hbar}{2m_e}, g_e = 2.$

$\mathbf{J} = \mathbf{L} + \mathbf{S}$ is the total angular momentum of the electron in an atom.

I is the nuclear spin angular momentum. The nuclear spin quantum number is determined by the specific atom in question, but in the case of hydrogen, it is the proton spin and has quantum number $\frac{1}{2}$. The magnetic moment is given by: $\mathbf{M}_I = \frac{g_p \mu_N}{\hbar}\mathbf{I}, \mu_N = \frac{e\hbar}{2M_p}, g_p \simeq 5.585.$

$\mathbf{F} = \mathbf{J} + \mathbf{I} = \mathbf{L} + \mathbf{S} + \mathbf{I}$ is the total angular momentum of the atom (in its own frame of reference).

The fine and hyperfine Hamiltonians:

The fine structure Hamiltonian is the leading order in α correction (specifically $\frac{H_f}{H_0} \sim \alpha^2$) to the Hamiltonian for an electron in a fixed Coulomb potential bound state, so $H = H_0 + H_f + \ldots$ (where $H_0 = \frac{p^2}{2m_e} + V(r) = \frac{p^2}{2m_e} - \frac{e^2}{r}$). It can be broken down into three terms:

$H_f = W_{mv} + W_{SO} + W_D$

$W_{mv} = -\frac{p^4}{8m_e^3 c^2}$ is the "variation of mass with velocity" term. It represents the leading order relativistic correction to the non-relativistic kinetic energy term:

$KE_{relativistic} = m_e c^2 + \frac{p^2}{2m_e} - \frac{p^4}{8m_e^3 c^2} + \ldots = const. + (H_0 - V(r)) + W_{mv} + \ldots$

$W_{SO} = \frac{1}{m_e^2 c^2}\frac{1}{r}\frac{dV(r)}{dr}\mathbf{L}\cdot\mathbf{S} = \frac{e^2}{m_e^2 c^2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S}$ is the "spin-orbit coupling" term. It represents the interaction of the magnetic moment of the electron spin with the magnetic field that arises in the electron's frame of reference as a result of motion in the Coulomb potential electric field.

$W_D = \frac{\hbar^2}{8m_e^2c^2}\nabla^2 V(r) = -\frac{\pi e^2\hbar^2}{2m_e^2c^2}\delta^3\left(\vec{r}\right)$ is the Darwin term. This term is the leading order correction from the Dirac equation, and manifests itself as a non-local interaction between the electron and the Coulomb field (despite the fact that the Dirac equation interaction is completely local).

The hyperfine structure Hamiltonian are the corrections that arise as a result of considering the nuclear spin and its resulting magnetic field. The hyperfine Hamiltonian is about three orders of magnitude smaller than fine structure Hamiltonian.

$$H_{hf} = -\frac{\mu_0}{4\pi}\left\{-\frac{e}{m_e r^3}\mathbf{L}\cdot\mathbf{M}_I + \frac{1}{r^3}[3(\mathbf{M}_S\cdot\mathbf{n})(\mathbf{M}_I\cdot\mathbf{n}) - \mathbf{M}_S\cdot\mathbf{M}_I] + \frac{8\pi}{3}\mathbf{M}_S\cdot\mathbf{M}_I\delta^3\left(\vec{r}\right)\right\}$$

The first term represents the interaction of the nuclear spin magnetic moment with the magnetic field created by the motion of the electron. The second term represents the dipole-dipole interaction between nuclear and electron spin magnetic moments. The third term represents the interaction of the electron spin magnetic moment with the magnetic field inside the proton.