Ph195a Midterm EXAM

Due Tuesday 11/6/01 at 5:00pm, at the Instructor's office (24 Bridge Annex).

This exam is closed book, but you may use one page (both sides of a single 8.5"x11" sheet of paper) of your own notes. **There is a time limit of three hours for this exam.**

1. For all parts of this problem let H be a Hilbert space spanned by the basis kets $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$, and let a and b be arbitrary complex constants.

(a) (5 points) Which of the following are Hermitian operators on H?

1. $|0\rangle\langle 1| + i|1\rangle\langle 0|$,
2. $|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|$,
3. $(a|0\rangle + |1\rangle)^\dagger (a|0\rangle + |1\rangle)$,
4. $(a|0\rangle + b^*|1\rangle)^\dagger (b|0\rangle - a^*|1\rangle)|2\rangle\langle 1| + |3\rangle\langle 3|$,
5. $|0\rangle\langle 0| + i|1\rangle\langle 0| - i|0\rangle\langle 1| + |1\rangle\langle 1|$.

(b) (5 points) Find the spectral decomposition of the following operator on H:

$$\mathbf{K} = |0\rangle\langle 0| + 2|1\rangle\langle 2| + 2|2\rangle\langle 1| - |3\rangle\langle 3|.$$

(c) (5 points) Let $|\Psi\rangle$ be a normalized ket in H, and let $\mathbf{1}$ denote the identity operator on H. Is the operator

$$\mathbf{B} \equiv \frac{1}{\sqrt{2}}(\mathbf{1} + |\Psi\rangle\langle\Psi|)$$

a projection operator?

(d) (5 points) Find the spectral decomposition of the operator **B** that was defined in part (c).

2. (10 points) Let $H_A = span\{|0_A\rangle, |1_A\rangle\}$ and $H_B = span\{|0_B\rangle, |1_B\rangle\}$ be two-dimensional Hilbert spaces, and let $|\Psi_{AB}\rangle$ be a factorizable state in the joint state space $H_A \otimes H_B$. Specify necessary and sufficient conditions on $|\Psi_{AB}\rangle$ such that $\mathbf{U}_{AB}|\Psi_{AB}\rangle$ is an entangled state, where

$$\mathbf{U}_{AB} = |0_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 0_B| - |1_A\rangle\langle 1_A| \otimes |1_B\rangle\langle 1_B|.$$

3. (10 points) Let $\boldsymbol{\rho}_{AB}$ be the density operator corresponding to a state on some joint Hilbert space $H_A \otimes H_B$. Show that the purity (trace of the square of) of a reduced density operator is **not** a good measure of entanglement if $\boldsymbol{\rho}_{AB}$ is allowed to be a mixed state.

4. (20 points) If we know that $\boldsymbol{\rho}_A$ and $\boldsymbol{\rho}_B$ are density operators such that

$$\mathbf{Tr}[\boldsymbol{\rho}_A \boldsymbol{\rho}_B] = 0,$$

what can we say about the possibility of finding a standard measurement that distinguishes perfectly between the corresponding mixed-state preparations?

5. Let Alice, Bob, and Charlie be in posession of quantum systems whose states live in $H_A = span\{|0_A\rangle, |1_A\rangle\}$, $H_B = span\{|0_B\rangle, |1_B\rangle\}$, and $H_C = span\{|0_C\rangle, |1_C\rangle\}$, respectively. Supose that the joint state of these systems has initially been prepared as the (three-way) entangled state

$$|\Psi_{ABC}\rangle = \frac{1}{\sqrt{2}}(|0_A 0_B 0_C\rangle + |1_A 1_B 1_C\rangle).$$

(a) (5 points) What is the reduced density operator on $H_A \otimes H_B$ if we take a partial trace over H_C?

(b) (5 points) Suppose Charlie performs a measurement specified by the partial projectors $\mathbf{1}^A \otimes \mathbf{1}^B \otimes |0_C\rangle\langle 0_C|$ and $\mathbf{1}^A \otimes \mathbf{1}^B \otimes |1_C\rangle\langle 1_C|$. Compute the probabilites of the possible outcomes, as well as the corresponding post-measurement states. Show that this ensemble is consistent with your answer from part (a).

(c) (5 points) Suppose Charlie performs a measurement specified by the partial projectors $\mathbf{1}^A \otimes \mathbf{1}^B \otimes |x_C\rangle\langle x_C|$ and $\mathbf{1}^A \otimes \mathbf{1}^B \otimes |y_C\rangle\langle y_C|$, where

$$|x_C\rangle = \frac{1}{\sqrt{2}}(|0_C\rangle + |1_C\rangle),$$

$$|y_C\rangle = \frac{1}{\sqrt{2}}(|0_C\rangle - |1_C\rangle).$$

Again compute the outcome probabilites and corresponding post-measurement states, and show that this ensemble is consistent with your answer from part (a).

(d) (5 points) Suppose Alice and Bob know that Charlie has performed one of the two measurements from parts (b) and (c), but they do not know which (assume equal probabilities) measurement he performed nor do they know the outcome. Write down the quantum ensemble that you think Alice and Bob should use to describe the post-measurement state on $H_A \otimes H_B$. Is this consistent with the reduced density operator from part (a)? How should Alice and Bob change their description of the post-measurement state if Charlie subsequently tells them which measurement he performed and what the outcome was?