

**Ph195b Midterm EXAM**

**Due Tuesday 12 February at 5:00pm, in the box outside 24 Bridge Annex.**

3 hour time limit, one page of your own notes only.

Coherent states of a harmonic oscillator can formally be generated by acting on the ground state with the displacement operator,

$$\mathbf{D}(\alpha) = \exp(\alpha \mathbf{a}^\dagger - \alpha^* \mathbf{a}).$$

Squeezed states of a harmonic oscillator can formally be generated by acting on the ground state with the squeezing operator,

$$\mathbf{S}(\xi) = \exp\left[\frac{1}{2}\xi^* \mathbf{a}^2 - \frac{1}{2}\xi (\mathbf{a}^\dagger)^2\right].$$

In what follows, you may find the following relations useful.

1. For operators  $\mathbf{A}$  and  $\mathbf{B}$  such that  $[\mathbf{A}, [\mathbf{A}, \mathbf{B}]] = [\mathbf{B}, [\mathbf{A}, \mathbf{B}]] = 0$ ,

$$\exp(\mathbf{A} + \mathbf{B} + [\mathbf{A}, \mathbf{B}]/2) = \exp(\mathbf{A}) \exp(\mathbf{B}).$$

2. For operators  $\mathbf{A}$  and  $\mathbf{B}$  such that  $[\mathbf{A}, [\mathbf{A}, \mathbf{B}]] = \beta \mathbf{B}$  with  $\beta$  a constant,

$$\exp(\lambda \mathbf{A}) \mathbf{B} \exp(-\lambda \mathbf{A}) = \mathbf{B} \cosh(\lambda \sqrt{\beta}) + \frac{[\mathbf{A}, \mathbf{B}]}{\sqrt{\beta}} \sinh(\lambda \sqrt{\beta}),$$

where  $\lambda$  must also be a constant.

3. For operators  $\mathbf{A}$  and  $\mathbf{B}$  such that  $[\mathbf{A}, \mathbf{B}] = \gamma \mathbf{B}$  with  $\gamma$  a constant,

$$\exp(\lambda \mathbf{A}) \mathbf{B} \exp(-\lambda \mathbf{A}) = \exp(\lambda \gamma) \mathbf{B},$$

where  $\lambda$  must also be a constant.

1. Let  $\xi = r \exp(2i\varphi)$ , where  $r$  and  $\varphi$  are positive real parameters.

(a) (20 points) Show that

$$\mathbf{S}^\dagger(\xi) \mathbf{a} \mathbf{S}(\xi) = \mathbf{a} \cosh r - \mathbf{a}^\dagger \exp(2i\varphi) \sinh r,$$

$$\mathbf{S}^\dagger(\xi) \mathbf{a}^\dagger \mathbf{S}(\xi) = \mathbf{a}^\dagger \cosh r - \mathbf{a} \exp(-2i\varphi) \sinh r.$$

(b) (20 points) Suppose we have an initial potential  $\mathbf{V}(\mathbf{x}) = \frac{1}{2}m\omega^2 \mathbf{x}^2$  and prepare the system in its ground state  $|0\rangle$ . If we suddenly switch the potential to  $\mathbf{V}'(\mathbf{x}) = \frac{1}{2}m(\omega')^2 \mathbf{x}^2$  we will (as discussed in the notes) produce a squeezed state, which may be written  $\mathbf{S}(\xi)|0'\rangle$  where  $|0'\rangle$  is the ground state of the new Hamiltonian. Using the results stated above, find a relation between  $\omega$ ,  $\omega'$ , and  $r$ . Hint: remember that  $\mathbf{x}$  is  $\mathbf{x}$  and  $\mathbf{p}$  is  $\mathbf{p}$ , regardless of what potential is applied.

2. (20 points) Consider the family of displaced squeezed states

$$|\alpha; \xi\rangle = \mathbf{D}(\alpha) \mathbf{S}(\xi) |0\rangle.$$

Assuming a fixed positive real value of  $\xi$ , find a closed expression for  $\langle \alpha_1; \xi | \alpha_2; \xi \rangle$ . Interpret your answer for the special cases where  $\alpha_1$  and  $\alpha_2$  are either both purely real or both purely imaginary.

3. (20 points) Consider a particle with two internal states that also moves in one kinetic dimension. Let the Hamiltonian for this system be

$$\mathbf{H} = \Pi_+ \otimes \mathbf{H}_+ + \Pi_- \otimes \mathbf{H}_-,$$

$$\mathbf{H}_+ = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}k_+\mathbf{x}^2,$$

$$\mathbf{H}_- = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}k_-\mathbf{x}^2,$$

where  $\Pi_+$  and  $\Pi_-$  are projectors onto the particle's internal states  $|+\rangle$  and  $|-\rangle$ , and  $k_{\pm}$  are positive real constants. Let the initial state be

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \otimes \mathbf{D}(\alpha)|0_+\rangle,$$

where  $|0_+\rangle$  is the ground state of  $\mathbf{H}_+$ . Compute

$$\langle \mathbf{x} \rangle_t \equiv \langle \Psi(t) | \mathbf{x} | \Psi(t) \rangle.$$

Hint: think about this problem before diving in to calculate.