Ph195 – study problems for 12/3/01

1NS. The Hamiltonian for a spin-$\frac{1}{2}$ particle in a static applied magnetic field $\vec{B} = B_0\hat{z}$ is simply

$$\mathbf{H}_0 = -\gamma B_0 \mathbf{S}_z.$$

Find a time-dependent operator

$$\mathbf{S}_u(t) = c_x(t)\mathbf{S}_x + c_y(t)\mathbf{S}_y$$

such that

$$\mathbf{T}(0,t)\,\mathbf{S}_u(t)\mathbf{T}(t,0) = \mathbf{S}_x,$$

where

$$\mathbf{T}(t,0) = \exp(-i\mathbf{H}_0\,t/\hbar)$$

is the time-development operator for $\mathbf{H}_0$. Infer that $\langle \mathbf{S}_u(t)\rangle$ is a constant.

2NS. Suppose the total Hamiltonian for a spin-$\frac{1}{2}$ particle is

$$\mathbf{H} = -\gamma[B_0\mathbf{S}_z + b_1(\cos(\omega t)\mathbf{S}_x + \sin(\omega t)\mathbf{S}_y)],$$

which includes a static field B_0 in the z direction plus a rotating field in the $x - y$ plane. Let the state of the particle be written

$$|\Psi(t)\rangle = a(t)|+_z\rangle + b(t)|-_z\rangle,$$

with normalization $|a|^2 + |b|^2 = 1$ and initial conditions

$$a(0) = 0, \quad b(0) = 1.$$

Show that

$$|a(t)|^2 = \frac{(\gamma b_1)^2}{\Delta^2 + (\gamma b_1)^2}\sin^2\left(\frac{t}{2}\sqrt{\Delta^2 + (\gamma b_1)^2}\right),$$

where $\Delta \equiv -\gamma B_0 - \omega$. This expression is known as the *Rabi Formula*.

3. The Master Equation (in the rotating frame) for a resonantly-driven, damped two-level system may be written

$$\frac{d}{dt}\boldsymbol{\rho}(t) = -\frac{i}{\hbar}[\mathbf{H},\boldsymbol{\rho}(t)] + \frac{1}{2}\Gamma(2\boldsymbol{\sigma}_-\boldsymbol{\rho}(t)\boldsymbol{\sigma}_+ - \boldsymbol{\sigma}_+\boldsymbol{\sigma}_-\boldsymbol{\rho}(t) - \boldsymbol{\rho}(t)\boldsymbol{\sigma}_+\boldsymbol{\sigma}_-),$$

where

$$\boldsymbol{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and the Hamiltonian (in the rotating frame) is simply

$$H = -\gamma b_1\mathbf{S}_x = -\frac{1}{2}\hbar\gamma b_1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here b_1 appears as a driving strength and Γ corresponds to the decay rate of the “excited” state $\langle +_z | = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

Find an exact solution for the steady-state density matrix $\boldsymbol{\rho}_{ss}$ in terms of the parameter $\eta \equiv \Gamma/(\gamma b_1)$. Steady-state is defined by

$$\frac{d}{dt}\boldsymbol{\rho}_{ss} = 0.$$

(Hint: you can use Hermiticity of $\boldsymbol{\rho}$ to simplify the equations you need to solve). Do not bother to transform back to the stationary frame. Plot the steady-state population inversion,

$$w = \frac{1}{2}\mathrm{Tr}[\boldsymbol{\sigma}_z \boldsymbol{\rho}_{ss}]$$

and the steady-state "dipole coherences"

$$u = \frac{1}{2}\mathrm{Tr}[\boldsymbol{\sigma}_x \boldsymbol{\rho}_{ss}],$$

$$v = \frac{1}{2}\mathrm{Tr}[\boldsymbol{\sigma}_y \boldsymbol{\rho}_{ss}],$$

as functions of η. Use a log scale for η and plot a range from $10^{-4} \le \eta \le 10^4$. Note that (u, v, w) are also the Cartesian components of the steady-state Bloch vector $\langle \vec{S} \rangle$.