Ph195b, lecture notes 1/28/02

Eigenfunctions of the Harmonic Oscillator

Recall from last time that the eigenstates of the linear harmonic oscillator (LHO) may be written

$$\psi_n(\xi) = C_n \exp(-\xi^2/2)\, v_e(\xi; n), \quad n \text{ even}$$
$$\psi_n(\xi) = C_n \exp(-\xi^2/2)\, v_o(\xi; n), \quad n \text{ odd}$$

where $\xi \equiv (m\omega/\hbar)^{1/2}\, x$, C_n is a normalization constant that we have yet to determine, and v_e and v_o are “variation functions” within the Gaussian envelope,

$$v_e(\xi; n) = 1 - \frac{2n}{2!}\xi^2 + \frac{2^2 n(n-2)}{4!}\xi^4 - \frac{2^3 n(n-2)(n-4)}{6!}\xi^6 + \cdots,$$
$$v_o(\xi; n) = \xi - \frac{2(n-1)}{3!}\xi^3 + \frac{2^2(n-1)(n-3)}{5!}\xi^5 - \cdots.$$

It turns out that v_e and v_o are equivalent, except for normalization, to the *Hermite polynomials*

$$H_n(\xi) = (-1)^{n/2}\frac{n!}{(n/2)!}v_e(\xi; n), \qquad n \text{ even}$$
$$H_n(\xi) = (-1)^{(n-1)/2}\frac{2(n!)}{[(n-1)/2]!}v_o(\xi; n), \quad n \text{ odd.}$$

The $H_n(\xi)$ are generated by the same recursion relation that generates v_e and v_o, except that we must take the initial coefficient (c_{n0} or c_{n1}) to be the given n-dependent prefactor rather than 1. With this normalization convention, which has the effect of setting the coefficient of the highest power of ξ to be 2^n, the Hermite polynomials have very compact and useful representations that can be used to derive matrix elements of various operators between the LHO eigenstates. Merzbacher lists some of the low-n Hermite polynomials explicitly, on page 84.

Note that since the H_n are proportional to our variation functions $v_{e/o}$ they satisfy the same differential equation

$$\frac{d^2 H_n(\xi)}{d\xi^2} - 2\xi\frac{dH_n(\xi)}{d\xi} + 2nH_n(\xi) = 0,$$

which is often taken (together with an appropriate normalization condition) as an implicit definition of the Hermite polynomials. Recall that in the case of the LHO eigenfunctions, we arrived at the corresponding differential equation for $v_{e/o}$ by inserting a Gaussian-envelope *ansatz* in the position-representation Schrödinger Equation.

The $H_n(\xi)$ can also be defined explicitly with the help of a *generating function,*

$$F(\xi, s) \equiv \sum_{n=0}^{\infty}\frac{H_n(\xi)}{n!}s^n.$$

Magically, we only need to know a few simple properties of the Hermite polynomials in order to derive (see below) a compact expression for this crazy object! From it, we can easily recover analytic expressions for the $H_n(\xi)$ themselves – assuming the validity of a Taylor expansion

$$F(\xi,s) = \sum_{n=0}^{\infty} \frac{s^n}{n!}\left(\frac{d^n}{ds^n}F(\xi,s)\right)_{s=0},$$

we obtain

$$H_n(\xi) = \left(\frac{d^n}{ds^n}F(\xi,s)\right)_{s=0}$$

simply by equating powers of s between the two above expressions for $F(\xi,s)$.

And now we move on to an exact expression for the generating function. It is easy to see that the H_n satisfy

$$\frac{dH_n(\xi)}{d\xi} = 2nH_{n-1}(\xi),$$

for instance by differentiating the above expressions for $v_{e/o}$ term-by-term to show that

$$\frac{dv_e(\xi;n)}{d\xi} = -2nv_o(\xi;n-1),$$
$$\frac{dv_o(\xi;n)}{d\xi} = v_e(\xi;n-1).$$

Applying the Hermite normalizations we then obtain

$$\begin{aligned}\frac{dH_n(\xi)}{d\xi} &= (-1)^{n/2}\frac{n!}{(n/2)!}\frac{dv_e(\xi;n)}{d\xi}, \qquad n \text{ even}\\ &= (-1)^{n/2}\frac{n!}{(n/2)!}(-2n)v_o(\xi;n-1)\\ &= 2n(-1)^{(n-2)/2}\frac{n[(n-1)!]}{(n/2)((n-2)/2)!}v_o(\xi;n-1)\\ &= 2nH_{n-1}(\xi),\end{aligned}$$

and similarly

$$\begin{aligned}\frac{dH_n(\xi)}{d\xi} &= (-1)^{(n-1)/2}\frac{2(n!)}{[(n-1)/2]!}\frac{dv_o(\xi;n)}{d\xi}, \qquad n \text{ odd}\\ &= (-1)^{(n-1)/2}\frac{2(n!)}{[(n-1)/2]!}v_e(\xi;n-1)\\ &= 2n(-1)^{(n-1)/2}\frac{(n-1)!}{[(n-1)/2]!}v_e(\xi;n-1)\\ &= 2nH_{n-1}(\xi).\end{aligned}$$

As a consequence, we find that

$$\begin{aligned}\frac{\partial F(\xi,s)}{\partial\xi} &= \sum_{n=1}^{\infty}\frac{2nH_{n-1}(\xi)}{n!}s^n\\ &= 2s\sum_{n=1}^{\infty}\frac{H_{n-1}(\xi)}{(n-1)!}s^{n-1}\\ &= 2sF(\xi,s).\end{aligned}$$

Hence we may formally integrate

$$\log[F(\xi,s)] = 2s\xi,$$
$$F(\xi,s) = F(0,s)\exp(2s\xi).$$

But we see from the original definition that

$$
\begin{aligned}
F(0,s) &= \sum_{n=0}^{\infty} \frac{H_n(0)}{n!} s^n \\
&= \sum_{n=0,2,4,\ldots}^{\infty} \frac{(-1)^{n/2}}{(n/2)!} s^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} s^{2n} \\
&= \exp(-s^2),
\end{aligned}
$$

and the generating function can be given explicitly as

$$
\begin{aligned}
F(\xi,s) &= \exp(2s\xi - s^2) \\
&= \exp\left(\xi^2 - (s-\xi)^2\right).
\end{aligned}
$$

Finally, we may write

$$
\begin{aligned}
H_n(\xi) &= \left(\frac{d^n}{ds^n} F(\xi,s)\right)_{s=0} \\
&= \left(\frac{d^n}{ds^n} \exp\left(\xi^2 - (s-\xi)^2\right)\right)_{s=0}.
\end{aligned}
$$

We next note that this is equivalent to

$$
\exp(\xi^2)\left(\frac{d^n}{ds^n} \exp\left(-(s-\xi)^2\right)\right)_{s=0}
$$

and apply a bit of trickery. Think of $\exp\left(-(s-\xi)^2\right)$ as a function of two variables – like a surface. Because s and $(-\xi)$ appear symmetrically, we know that the derivatives (slope, curvature, etc.) at any point on this surface will be symmetric with respect to s and $(-\xi)$. Hence,

$$
\begin{aligned}
\frac{d^n}{ds^n} \exp\left(-(s-\xi)^2\right) &= \frac{d^n}{d(-\xi)^n} \exp\left(-(s-\xi)^2\right) \\
&= (-1)^n \frac{d^n}{d\xi^n} \exp\left(-(s-\xi)^2\right),
\end{aligned}
$$

so

$$
\begin{aligned}
H_n(\xi) &= \exp(\xi^2)\left(\frac{d^n}{ds^n} \exp\left(-(s-\xi)^2\right)\right)_{s=0} \\
&= (-1)^n \exp(\xi^2) \frac{d^n}{d\xi^n} \exp(-\xi^2).
\end{aligned}
$$

This is now a handy and very cool expression for the $H_n(\xi)$. You can see how finite-order polynomials are generated via product rule for differentiation, and in the end the two exponentials cancel through. Merzbacher shows on pages 86-87 that it follows from this definition of the Hermite polynomials that all n roots of $H_n(\xi)$ lie on the real axis. Hence, $\psi_n(x)$ must have n nodes placed symmetrically about the origin ($x = 0$), with either a local extremum (n even) or node (n odd) at the origin.

As promised, we now put the generating function to work. For us, its principle utility will be in computing moments of the position operator $\hat{x}$ for LHO eigenstates, as well as matrix elements of powers of $\hat{x}$ between different eigenstates. Consider the general matrix element

$$\langle \Psi_n | \hat{x}^p | \Psi_k \rangle \leftrightarrow \int_{-\infty}^{+\infty} dx \, \psi_n^*(x) \, x^p \, \psi_k(x)$$
$$= C_n^* C_k (\hbar/m\omega)^{(p+1)/2} \int_{-\infty}^{+\infty} d\xi \, \tilde{\psi}_n^*(\xi) \tilde{\psi}_k(\xi) \, \xi^p,$$

where we have included complex conjugations even though we know the eigenfunctions are real, and

$$\tilde{\psi}_n(\xi) \equiv \psi_n(\xi)/C_n.$$

It turns out that we can use the Hermite polynomial generating function to derive a general expression for the value of the integral

$$I_{nkp} = \int_{-\infty}^{+\infty} d\xi \, \tilde{\psi}_n^*(\xi) \tilde{\psi}_k(\xi) \, \xi^p$$
$$= \int_{-\infty}^{+\infty} d\xi \, H_n(\xi) H_k(\xi) \exp(-\xi^2) \, \xi^p \quad (n, k, p \geq 0).$$

To do so we must first construct a different integral,

$$\int_{-\infty}^{+\infty} d\xi \, \exp\left(\xi^2 - (s-\xi)^2\right) \exp\left(\xi^2 - (t-\xi)^2\right) \exp(2\lambda\xi - \xi^2)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{s^n t^k}{n! k!} \int_{-\infty}^{+\infty} d\xi \, H_n(\xi) H_k(\xi) \exp(2\lambda\xi - \xi^2)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{s^n t^k (2\lambda)^p}{n! k! p!} \int_{-\infty}^{+\infty} d\xi \, H_n(\xi) H_k(\xi) \exp(-\xi^2) \, \xi^p$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{s^n t^k (2\lambda)^p}{n! k! p!} I_{nkp}.$$

where we have used the definition of the Hermite generating function in going from the first to the second line, and a Taylor expansion of the exponential in going from the second to the third. It turns out that the original Gaussian-looking integral (on the first line) can be done exactly, yielding

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{s^n t^k (2\lambda)^p}{n! k! p!} I_{nkp} = \pi^{1/2} \exp(\lambda^2 + 2(st + \lambda s + \lambda t)).$$

From this magic equation we can (in principle) extract the values of I_{nkp} by equating coefficients in a power-series expansion of the exponential. In practice this can require some finesse.

Writing things out a bit more explicitly,

$$\exp(\lambda^2 + 2(st + \lambda s + \lambda t)) = \sum_{m=0}^{\infty} \frac{1}{m!} (\lambda^2 + 2(st + \lambda s + \lambda t))^m.$$

One thing we can do is isolate the terms with $p = 0$, which will give us the inner products between eigenfunctions, including (finally!) their normalizations. To proceed we do a bit of regrouping and then apply a binomial expansion,

$$\sum_{m=0}^{\infty} \frac{1}{m!} (\lambda^2 + 2(st + \lambda s + \lambda t))^m = \sum_{m=0}^{\infty} \frac{1}{m!} ([\lambda^2 + 2(\lambda s + \lambda t)] + 2st)^m$$
$$= \sum_{m=0}^{\infty} \sum_{r=0}^{m} \frac{1}{m!} \binom{m}{r} (2st)^r [\lambda^2 + 2(\lambda s + \lambda t)]^{m-r}.$$

Clearly only terms with $r = m$ will contribute to I_{nk0} since the power of λ must end up being zero. This makes the binomial coefficient one, and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{s^n t^k}{n!k!}I_{nk0} = \pi^{1/2}\sum_{m=0}^{\infty}\frac{1}{m!}(2st)^m.$$

Hence

$$I_{nk0} = \int_{-\infty}^{+\infty} d\xi\,\psi_n^*(\xi)\psi_k(\xi) = 0$$

unless $n = k$, which verifies orthogonality of the LHO eigenfunctions. When $n = k$, we find

$$\frac{I_{nn0}}{(n!)^2} = (n!)^{-2}C_n^{-2}(\hbar/m\omega)^{-1/2}\langle\Psi_n|\Psi_n\rangle$$
$$= \pi^{1/2}\frac{2^n}{n!}.$$

Hence,

$$C_n^2 = \frac{(m\omega/\hbar)^{1/2}}{\pi^{1/2}2^n n!},$$
$$C_n = \frac{(m\omega/\pi\hbar)^{1/4}}{(2^n n!)^{1/2}},$$

and we have at last a normalized expression for the LHO eigenstates

$$\psi_n(x) = 2^{-n/2}(n!)^{-1/2}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)H_n\left((m\omega/\hbar)^{1/2}x\right).$$

Please keep in mind that the rightmost expression in parentheses here is the argument of the Hermite polynomial, not an extra multiplicative term!

We can similarly compute matrix elements of x between LHO eigenstates,

$$\langle\Psi_n|\hat{x}|\Psi_k\rangle = \frac{(m\omega/\hbar)^{-1/2}}{\pi^{1/2}(2^n n!)^{1/2}(2^k k!)^{1/2}}I_{nk1}.$$

We proceed from an expression we derived for the $p = 0$ case above,

$$\sum_{m=0}^{\infty}\frac{1}{m!}(\lambda^2+2(st+\lambda s+\lambda t))^m = \sum_{m=0}^{\infty}\sum_{r=0}^{m}\frac{1}{m!}\binom{m}{r}(2st)^r[\lambda^2+2(\lambda s+\lambda t)]^{m-r}.$$

Now for $p = 1$, we will want to consider only the case $r = m-1$ (the $m = 0$ case clearly does not contribute). This makes the binomial coefficient equal to m, and we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{s^n t^k 2\lambda}{n!k!}I_{nk1} = \pi^{1/2}\sum_{m=1}^{\infty}\frac{1}{(m-1)!}(2st)^{m-1}2(\lambda s+\lambda t).$$

Clearly we have $I_{nk1} = 0$ unless n and k differ by exactly one. We can write explicitly,

$$
\begin{aligned}
I_{nk1} &= \pi^{1/2}[2^{k-1}(k!)\,\delta_{n,k-1} + 2^{n-1}(n!)\,\delta_{k,n-1}],\\
\langle \Psi_n|\hat{x}|\Psi_k\rangle &= \frac{(m\omega/\hbar)^{-1/2}}{(2^n n!)^{1/2}(2^k k!)^{1/2}}[2^{k-1}(k!)\,\delta_{n,k-1} + 2^{n-1}(n!)\,\delta_{k,n-1}]\\
&= \frac{(m\omega/\hbar)^{-1/2}}{(2^{n+k} n!\,k!)^{1/2}}[2^{k-1}(k!)\,\delta_{n,k-1} + 2^{n-1}(n!)\,\delta_{k,n-1}]\\
&= (\hbar/m\omega)^{1/2}\left[\left(\frac{k}{2}\right)^{1/2}\delta_{n,k-1} + \left(\frac{n}{2}\right)^{1/2}\delta_{k,n-1}\right]\\
&= (\hbar/m\omega)^{1/2}\left[\left(\frac{n+1}{2}\right)^{1/2}\delta_{k,n+1} + \left(\frac{n}{2}\right)^{1/2}\delta_{k,n-1}\right].
\end{aligned}
$$

A similar computation can be carried out for $p = 2$, *et cetera*.

It is important to note that we can say a lot about matrix elements of $\hat{x}^p$ based on symmetry arguments alone. Recall that an LHO eigenfunction $\psi_n(x)$ is even with respect to reflection through the origin (parity symmetry) when n is even, and odd when n is odd. Hence if we consider the general matrix element

$$
\langle \Psi_n|\hat{x}^p|\Psi_k\rangle = \int_{-\infty}^{+\infty} dx\, x^p\, \psi_n(x)\psi_k(x),
$$

we see the integrand will have a well-defined overall parity symmetry (since each of its factors does). So if p is odd, either n or k must be odd (but not both!) in order for the matrix element to be non-zero. If p is even, then n and k must either both be even or both odd. Based on the calculations we did above for I_{nk0} and I_{nk1}, we might start to suspect that p might even fix the exact difference between n and k? We'll see more about this when we introduce creation and annihilation operators below.

Merzbacher (on page 88) also discusses an integral representation for the Hermite polynomials,

$$
H_n(\xi) = \frac{2^n}{\pi^{1/2}}\int_{-\infty}^{+\infty} du\,(\xi + iu)^n \exp(-u^2).
$$

An important feature of this representation (which we'll not need otherwise) is that it can be used to derive the closure relation for the LHO eigenfunctions,

$$
\sum_{n=0}^{\infty} \psi_n^*(x')\psi_n(x) = \delta(x - x').
$$

In Dirac notation,

$$
\sum_{n=0}^{\infty} |\Psi_n\rangle\langle\Psi_n| = \mathbf{1}.
$$

This shows that the LHO eigenfunctions form a complete set. That is, the wave-function corresponding to any vector in the infinite-dimensional Hilbert space can be decomposed as a sum over LHO eigenstates. In a sense this should not be surprising (by analogy with finite-dimensional systems), but at the same time it's somewhat amazing to me that a discrete sum of smooth functions can reproduce a delta-function.

Motion of wave packets

Let's say we have been given a wave-function $\psi(x,0)$ corresponding to the initial state (at time $t = 0$) of an LHO. We know that we can compute (by inner products) the coefficients for an expansion in the LHO eigenbasis, so we may write

$$\psi(x,0) = \sum_{n=0}^{\infty} c_n \psi_n(x)$$

We found earlier that the energy of the nth eigenstate is

$$\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right),$$

so the time-evolution of the wave packet should be

$$\psi(x,t) = \exp(-i\omega t/2) \sum_{n=0}^{\infty} c_n \exp(-in\omega t)\, \psi_n(x).$$

In order to try to understand what this tells us, we can compute the evolution of the mean value of the position operator,

$$\begin{aligned}
\langle \hat{x} \rangle_t &= \int_{-\infty}^{+\infty} dx\, x |\psi(x,t)|^2 \\
&= \sum_{m,n=0}^{\infty} \int_{-\infty}^{+\infty} dx\, x\, c_n^* c_m \exp(-i(m-n)\omega t)\, \psi_n^*(x) \psi_m(x) \\
&= \sum_{m,n=0}^{\infty} c_n^* c_m \exp(-i(m-n)\omega t) \int_{-\infty}^{+\infty} dx\, x\, \psi_n^*(x) \psi_m(x) \\
&= \sum_{m,n=0}^{\infty} c_n^* c_m \exp(-i(m-n)\omega t) \langle \Psi_n | \hat{x} | \Psi_m \rangle.
\end{aligned}$$

Using the result we derived above for the matrix element,

$$\begin{aligned}
\langle \hat{x} \rangle_t &= \sum_{m,n=0}^{\infty} c_n^* c_m \exp(-i(m-n)\omega t)\, (\hbar/m\omega)^{1/2} \left[\left(\frac{n+1}{2}\right)^{1/2} \delta_{m,n+1} + \left(\frac{n}{2}\right)^{1/2} \delta_{m,n-1} \right] \\
&= (\hbar/2m\omega)^{1/2} \sum_{n=1}^{\infty} n^{1/2} [c_n^* c_{n-1} \exp(+i\omega t) + c_{n-1}^* c_n \exp(-i\omega t)].
\end{aligned}$$

Note that we can in fact restore the $n = 0$ lower limit without changing anything. If we define

$$c_n = |c_n| \exp(i\phi_n),$$

we arrive at

$$\langle \hat{x} \rangle_t = (2\hbar/m\omega)^{1/2} \sum_{n=0}^{\infty} n^{1/2} |c_n||c_{n-1}| \cos(\omega t + \phi_{n-1} - \phi_n).$$

Hence $\langle \hat{x} \rangle_t$ is given by the weighted sum of oscillating terms, all of which have exactly the same frequency ω but varying phases $(\phi_{n-1} - \phi_n)$. Such a sum necessarily oscillates harmonically at frequency ω as well, reflecting a strong dynamical analogy between the quantum and classical LHO's. Note that the overall amplitude of oscillation depends on the "distribution" of n, since the coefficients $|c_n|$ are bounded by overall normalization. Higher-energy components lead to larger amplitude of oscillation, and more than one non-zero $|c_n|$ is required for oscillatory motion. For a given energy distribution, the oscillation amplitude will be maximized if the components contribute “coherently,” that is, with *constant* phase $(\phi_{n-1} - \phi_n)$. This general feature of the LHO will soon lead us to the investigation of what are called “coherent states,” but please be careful to note that this nomenclature is not meant to imply that other sorts of states we will consider (such as the energy eigenstates) are necessarily “incoherent” in the sense of a density matrix!

Operator formalism for the LHO

We now turn to the algebraic method of deriving the LHO eigenspectrum. Don’t get mad when you see how much easier it is, because it really is important to see everything done both ways.

The basis of this method is to utilize the (non-Hermitian) *annihilation operator*

$$\mathbf{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\mathbf{x} + i\frac{\mathbf{p}}{m\omega}\right),$$

and its conjugate, the *creation operator*

$$\mathbf{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\mathbf{x} - i\frac{\mathbf{p}}{m\omega}\right).$$

Note that we may simply compute the commutator of $\mathbf{a}$ and $\mathbf{a}^\dagger$ since we already know that $[\mathbf{x}, \mathbf{p}] = i\hbar$,

$$\begin{aligned}
\left[\mathbf{a}, \mathbf{a}^\dagger\right] &= \sqrt{\frac{m\omega}{2\hbar}}\left(\left[\mathbf{x}, \mathbf{a}^\dagger\right] + \frac{i}{m\omega}\left[\mathbf{p}, \mathbf{a}^\dagger\right]\right) \\
&= \frac{m\omega}{2\hbar}\left([\mathbf{x}, \mathbf{x}] - \frac{i}{m\omega}[\mathbf{x}, \mathbf{p}] + \frac{i}{m\omega}[\mathbf{p}, \mathbf{x}] + \frac{1}{m\omega^2}[\mathbf{p}, \mathbf{p}]\right) \\
&= \frac{1}{2\hbar}(\hbar + \hbar) \\
&= 1.
\end{aligned}$$

Nice and simple! From this we note that

$$\begin{aligned}
\left[\mathbf{a}, \mathbf{a}^\dagger\right] &= \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = \mathbf{1}, \\
\mathbf{a}\mathbf{a}^\dagger &= \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}.
\end{aligned}$$

Next we need to re-express the LHO Hamiltonian in terms of $\mathbf{a}$ and $\mathbf{a}^\dagger$. Towards that end, we derive

$$\begin{aligned}
\mathbf{x} &= \sqrt{\frac{\hbar}{2m\omega}}\left(\mathbf{a} + \mathbf{a}^\dagger\right), \\
\mathbf{x}^2 &= \frac{\hbar}{2m\omega}\left[\mathbf{a}^2 + \mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a} + \left(\mathbf{a}^\dagger\right)^2\right], \\
\mathbf{p} &= -i\sqrt{\frac{m\hbar\omega}{2}}\left(\mathbf{a} - \mathbf{a}^\dagger\right), \\
\mathbf{p}^2 &= -\frac{m\hbar\omega}{2}\left[\mathbf{a}^2 - \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} + \left(\mathbf{a}^\dagger\right)^2\right].
\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{H} &= \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{x}^2 \\ &= -\frac{\hbar\omega}{4}\left[\mathbf{a}^2 - \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} + \left(\mathbf{a}^\dagger\right)^2\right] + \frac{\hbar\omega}{4}\left[\mathbf{a}^2 + \mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a} + \left(\mathbf{a}^\dagger\right)^2\right] \\ &= \frac{1}{2}\hbar\omega\left(\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a}\right) \\ &= \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right).\end{aligned}$$

This form of the Hamiltonian clearly shows the importance of the operator $\mathbf{a}^\dagger\mathbf{a}$, as we see that the energy eigenvalues will be proportional to its eigenvalues plus one-half. Writing the eigenvalue equation, in order to establish notation,

$$\mathbf{a}^\dagger\mathbf{a}|n\rangle = \lambda_n|n\rangle.$$

We can immediately show that the eigenvalues λ_n must be non-negative. To do so we consider the norm of the state obtained by letting $\mathbf{a}$ act on an arbitrary eigenvector $|n\rangle$, which must be a non-negative number:

$$(\mathbf{a}|n\rangle)^\dagger(\mathbf{a}|n\rangle) = \langle n|\mathbf{a}^\dagger\mathbf{a}|n\rangle = \lambda_n.$$

Hence λ_n cannot be negative.

Next we will show that once we have found any one eigenstate of $\mathbf{a}^\dagger\mathbf{a}$, the creation and annihilation operators can be used to obtain all the others. In particular, if $|n\rangle$ is an eigenstate, then so is $\mathbf{a}^\dagger|n\rangle$:

$$\begin{aligned}\left(\mathbf{a}^\dagger\mathbf{a}\right)\mathbf{a}^\dagger|n\rangle &= \mathbf{a}^\dagger\left(\mathbf{a}\mathbf{a}^\dagger\right)|n\rangle \\ &= \mathbf{a}^\dagger\left(\mathbf{a}^\dagger\mathbf{a} + 1\right)|n\rangle \\ &= (\lambda_n + 1)\mathbf{a}^\dagger|n\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\left(\mathbf{a}^\dagger\mathbf{a}\right)\mathbf{a}|n\rangle &= \left(\mathbf{a}\mathbf{a}^\dagger - 1\right)\mathbf{a}|n\rangle \\ &= \mathbf{a}\mathbf{a}^\dagger\mathbf{a}|n\rangle - \mathbf{a}|n\rangle \\ &= (\lambda_n - 1)\mathbf{a}|n\rangle.\end{aligned}$$

So all that remains is for us to find one eigenstate to get us started!

We can do this by noting that the annihilation operator takes any given eigenstate to one with an eigenvalue decreased by one, but we know that no eigenvalue of $\mathbf{a}^\dagger\mathbf{a}$ can be less than zero. Hence, there must exist a lowest eigenstate $|0\rangle$ such that

$$\mathbf{a}|0\rangle = 0.$$

If we now ask about the eigenvalue of this state,

$$\begin{aligned}\mathbf{a}^\dagger\mathbf{a}|0\rangle &= \mathbf{a}^\dagger(\mathbf{a}|0\rangle) \\ &= 0.\end{aligned}$$

Hence, $\lambda_0 = 0$. Then $\mathbf{a}^\dagger|0\rangle \propto |1\rangle$ is an eigenstate with eigenvalue $\lambda_1 = 1$, $\mathbf{a}^\dagger|1\rangle \propto (\mathbf{a}^\dagger)^2|0\rangle \propto |2\rangle$ has eigenvalue $\lambda_2 = 2$, *et cetera*. For this reason, $\mathbf{a}^\dagger\mathbf{a}$ is referred to as the number operator and is often denoted $\mathbf{N}$. So we now see that the eigenspectrum of $\mathbf{a}^\dagger\mathbf{a}$ is discrete, taking on values $\lambda_n = 0, 1, 2, \ldots$ (non-negative integers). Remind you of anything? It should remind of you of the non-negative integer n that appeared in our recursion relation fo the Hermite polynomials!

Returning to the LHO Hamiltonian, we thus find that

$$\mathbf{H} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right) = \hbar\omega\left(\mathbf{N} + \frac{1}{2}\right)$$

has energy eigenvalues

$$\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \ldots$$

Next we would like to see if we can also recover the LHO eigenfunctions by this algebraic method.

LHO eigenstates via operator algebra

Recall that in the 'brute force' approach, we determined eigenfunctions of the LHO by solving

$$\mathbf{H}|\Psi_n\rangle = \varepsilon_n|\Psi_n\rangle$$

as a differential equation in position representation. We first tried plugging in a power-series expansion for the eigenfunction, and obtained from this a three-term recursion relation that we decided was too hard to work with. We next tried a Gaussian-envelope *ansatz* and obtained a much nicer two-term recursion relation for coefficients in the power-series expansion of the 'variation function.' By examining convergence of this expansion we were able to show that the energy eigenspectrum must be proportional to the set of non-negative integers, and as a consequence the variation functions turn out to be finite polynomials of definite parity.

We then found that it was much easier (or at least faster) to derive the eigenspectrum using annihilation and creation operators. In particular, we showed just using Dirac algebra that the number operator $\mathbf{a}^\dagger\mathbf{a}$ has non-negative integers n as its eigenvalues

$$\mathbf{a}^\dagger\mathbf{a}|n\rangle = n|n\rangle,$$

and that the LHO energy eigenvalues followed from

$$\mathbf{H} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right),$$

$$\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

As an encore, we would now like to recover the wave functions corresponding to LHO eigenstates via similar algebraic techniques.

From the Dirac point of view, these wave functions appear as infinite 'vectors' of inner-products between the LHO eigenstates $|\Psi_n\rangle = |n\rangle$ and eigenstates $|x\rangle$ of the position operator:

$$\psi_n(x) = \langle x|n\rangle.$$

So far, the only thing we know that relates position to n is the operator equivalence

$$\mathbf{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\mathbf{a} + \mathbf{a}^\dagger\right).$$

As we saw last time, the annihilation and creation operators have a simple action in the $|n\rangle$ basis. Therefore, $\mathbf{a}$, $\mathbf{a}^\dagger$, and $\mathbf{x}$ should all have simple *matrix representations* in the LHO eigenbasis, and we might hope to be able to 'diagonalize' **x** to obtain vectors

$$\begin{bmatrix} \langle 0|x\rangle \\ \langle 1|x\rangle \\ \langle 2|x\rangle \\ \vdots \end{bmatrix} = \begin{bmatrix} \psi_0^*(x) \\ \psi_1^*(x) \\ \psi_2^*(x) \\ \vdots \end{bmatrix}$$

representing the position eigenstates. Of course, these will be infinite (do you see why?) but let's plunge ahead anyway.

First we need to determine the matrix representations of **a** and $\mathbf{a}^\dagger$. Remeber that last time we showed

$$\mathbf{a}|n\rangle \propto |n-1\rangle,$$

hence the only non-zero matrix elements of **a** will lie just above the diagonal. Writing

$$\langle n-1|\mathbf{a}|n\rangle = C_n,$$

we can compute

$$\begin{aligned} |C_n|^2 &= \langle n|\mathbf{a}^\dagger|n-1\rangle\langle n-1|\mathbf{a}|n\rangle \\ &= \langle n|\mathbf{a}^\dagger|n-1\rangle\langle n-1|\mathbf{a}|n\rangle + \sum_{m\neq n-1}\langle n|\mathbf{a}^\dagger|m\rangle\langle m|\mathbf{a}|n\rangle \\ &= \langle n|\mathbf{a}^\dagger\left(\sum_{m=0}^{\infty}|m\rangle\langle m|\right)\mathbf{a}|n\rangle \\ &= \langle n|\mathbf{a}^\dagger\mathbf{a}|n\rangle \\ &= n, \end{aligned}$$

where we have made use of the closure relation over LHO eigenstates. Hence

$$\mathbf{a}|n\rangle = \sqrt{n}\,|n-1\rangle,$$

where can have aribtrarily set the phase of C_n to zero since there are no particular constraints (other than consistency) on how we choose it. The matrix representation of the annihilation and creation operators are thus

$$\mathbf{a} \to \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & \sqrt{3} & \\ 0 & 0 & 0 & 0 & \\ & & & & \ddots \end{pmatrix},$$

$$\mathbf{a}^\dagger \to \begin{pmatrix} 0 & 0 & 0 & 0 & \\ \sqrt{1} & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & 0 & \\ 0 & 0 & \sqrt{3} & 0 & \\ & & & & \ddots \end{pmatrix},$$

and of the position operator

$$\mathbf{x} \to \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 1 & 0 & \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \\ & & & & \ddots \end{pmatrix}.$$

In the LHO eigenbasis then, the position eigenstates $|x\rangle$ satisfy

$$\sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 1 & 0 & \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} = x \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}.$$

This leads to the sequence of algebraic constraints

$$\begin{aligned} c_1 &= \sqrt{\frac{2m\omega}{\hbar}}\, x c_0, \\ c_0 + \sqrt{2}\, c_2 &= \sqrt{\frac{2m\omega}{\hbar}}\, x c_1, \\ \sqrt{2}\, c_1 + \sqrt{3}\, c_3 &= \sqrt{\frac{2m\omega}{\hbar}}\, x c_2, \\ &\vdots \\ \sqrt{n}\, c_{n-1} + \sqrt{n+1}\, c_{n+1} &= \sqrt{\frac{2m\omega}{\hbar}}\, x c_n. \end{aligned}$$

Recalling

$$c_n = \langle n | x \rangle = \psi_n^*(x),$$

we see that these constraints actually give us a recursion relation

$$\begin{aligned} \psi_1(x) &= \sqrt{\frac{2m\omega}{\hbar}}\, x \psi_0(x), \\ \psi_{n+1}(x) &= \frac{1}{\sqrt{n+1}} \left[\sqrt{\frac{2m\omega}{\hbar}}\, x \psi_n(x) - \sqrt{n}\, \psi_{n-1}(x) \right] \quad (n \geq 1) \end{aligned}$$

that allows us to determine all eigenfunctions with $n \geq 1$ in terms of $\psi_0(x)$. Magic! All we have to do now is figure out $\psi_0(x)$, which is easily done by solving

$$\begin{aligned} \mathbf{a}|0\rangle &= \sqrt{\frac{m\omega}{2\hbar}} \left(\mathbf{x} + i \frac{\mathbf{p}}{m\omega} \right) |0\rangle \\ &= 0 \end{aligned}$$

in the position representation. The differential equation is simply

$$\begin{aligned} \left(x + \frac{\hbar}{m\omega} \frac{d}{dx} \right) \psi_0(x) &= 0, \\ \frac{d}{dx} \psi_0(x) &= -\frac{m\omega}{\hbar} x \psi_0(x). \end{aligned}$$

The solution is easily guessed as

$$\psi_0(x) = C_0 \exp\left(-\frac{m\omega}{2\hbar}x^2\right),$$

a nice Gaussian function whose normalization we know to be

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

All other eigenfunctions can be determined by applying the recursion derived above.

You may be wondering how to get Hermite polynomials back into the story, though? One way is to note that the constraint we derived,

$$\sqrt{n}\, c_{n-1} + \sqrt{n+1}\, c_{n+1} = \sqrt{\frac{2m\omega}{\hbar}}\, x c_n,$$

can be turned into a known recursion relation for the Hermite polynomials,

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

by assigning

$$c_n = \psi_n^*(x) \propto 2^{-(n/2)}(n!)^{-1/2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right).$$

The important 'constant' of proportionality here turns out to be a function of x, as must be the case since for $n = 0$ we must have $c_0 = \psi_0^*(x)$ (and $H_0(x) = 1$). Hence

$$\psi_n(x) = 2^{-(n/2)}(n!)^{-1/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right).$$

Coherent and squeezed states of the LHO

So far we have seen several methods for deriving eigenvalues and eigenfunctions of the LHO. However, early on in the discussion we noted that these so-called 'harmonic oscillator eigenstates' don't actually oscillate harmonically. From the general expression

$$\langle \mathbf{x} \rangle_t = \sqrt{\frac{2\hbar}{m\omega}} \sum_{n=0}^{\infty} \sqrt{n}\, |c_n| |c_{n+1}| \cos(\omega t + \phi_{n-1} - \phi_n),$$

where

$$|\Psi(0)\rangle = \sum_{n=0}^{\infty} |c_n| \exp(i\phi_n) |n\rangle,$$

we see that a superpositions of different eigenstates is necessary to obtain sinusoidal variation of position. Of course, this has to be the case since eigenstates are stationary states of the Schrödinger Equation – operator moments are not allowed to evolve when the system is prepared initially in an energy eigenstate! Still, we may start to suspect that the LHO eigenstates are particularly boring from a dynamical point of view. Since we saw that the position operator is purely off-diagonal in the energy basis, we have $\langle \mathbf{x} \rangle = 0$ for every eigenstate. Likewise, $\langle \mathbf{p} \rangle = 0$ since $\mathbf{p} \propto \mathbf{a} - \mathbf{a}^\dagger$. So even though a state $|n\rangle$ with $n \gg 1$ somehow contains a lot of energy ($\Delta\mathbf{x}$ and $\Delta\mathbf{p}$ are $\gg 0$), the various position and momentum components don't actually manage to add up 'coherently' in such a way that an identifiable oscillation takes place!

Our task in this section of the notes will be to show that despite these highly non-classical properties of the LHO eigenstates, there do exist *superpositions* of LHO eigenstates with reassuringly classical behavior.

For a classical (undamped, undriven) harmonic oscillator, the equation of motion is simply

$$m\ddot{x} = -kx,$$

or

$$\ddot{x} = -\omega^2 x$$

with $\omega \equiv \sqrt{k/m}$. The solution is clearly

$$x(t) = A\exp(+i\omega t) + B\exp(-i\omega t),$$

subject to the initial conditions on both $x(0)$ and $p(0)$:

$$x(0) = A + B = x_0,$$

$$\left(\frac{dx}{dt}\right)_{t=0} = i\omega A - i\omega B = \frac{p_0}{m}.$$

Hence,

$$x(t) = \frac{1}{2}\left(x_0 - i\frac{p_0}{m\omega}\right)\exp(+i\omega t) + \frac{1}{2}\left(x_0 + i\frac{p_0}{m\omega}\right)\exp(-i\omega t)$$

$$= x_0\cos(\omega t) + \frac{p_0}{m\omega}\sin(\omega t).$$

It seems reasonable to ask if the quantum LHO admits time-dependent solutions with

$$\langle \mathbf{x} \rangle_t = \langle \mathbf{x} \rangle_0 \cos(\omega t) + \frac{\langle \mathbf{p} \rangle_0}{m\omega}\sin(\omega t)$$

$$= \frac{1}{2}\left(\langle \mathbf{x} \rangle_0 - i\frac{\langle \mathbf{p} \rangle_0}{m\omega}\right)\exp(+i\omega t) + \frac{1}{2}\left(\langle \mathbf{x} \rangle_0 + i\frac{\langle \mathbf{p} \rangle_0}{m\omega}\right)\exp(-i\omega t).$$

In order to really mimic classical behavior, we might also hope for the existence of states in which $\Delta\mathbf{x}$ remains small at all times. We know this isn't possible for a free particle (spreading of the wave packet), but the LHO has a confining potential so maybe we'll get lucky.

In order to try to derive such states, let's pull an old trick out of our first-term hat. Using the time-development operator formalism

$$|\Psi(t)\rangle = \exp(-i\mathbf{H}t/\hbar)|\Psi(0)\rangle,$$

we see that

$$\langle \mathbf{x} \rangle_t = \langle \Psi(t)|\mathbf{x}|\Psi(t)\rangle$$

$$= \langle \Psi(0)|\exp(+i\mathbf{H}t/\hbar)\,\mathbf{x}\,\exp(-i\mathbf{H}t/\hbar)|\Psi(0)\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\begin{array}{c} \langle \Psi(0)|\exp(+i\mathbf{H}t/\hbar)\,\mathbf{a}\,\exp(-i\mathbf{H}t/\hbar)|\Psi(0)\rangle + \\ \langle \Psi(0)|\exp(+i\mathbf{H}t/\hbar)\,\mathbf{a}^\dagger\,\exp(-i\mathbf{H}t/\hbar)|\Psi(0)\rangle \end{array}\right].$$

Comparing this with the classical equations above, we see that we need

$$\sqrt{\frac{\hbar}{2m\omega}}\langle\Psi(0)\,|\,\exp(+i\mathbf{H}t/\hbar)\,\mathbf{a}\,\exp(-i\mathbf{H}t/\hbar)\,|\Psi(0)\rangle = \frac{1}{2}\left(\langle\mathbf{x}\rangle_0 + i\frac{\langle\mathbf{p}\rangle_0}{m\omega}\right)\exp(-i\omega t),$$
$$\sqrt{\frac{\hbar}{2m\omega}}\langle\Psi(0)\,|\,\exp(+i\mathbf{H}t/\hbar)\,\mathbf{a}^\dagger\,\exp(-i\mathbf{H}t/\hbar)\,|\Psi(0)\rangle = \frac{1}{2}\left(\langle\mathbf{x}\rangle_0 - i\frac{\langle\mathbf{p}\rangle_0}{m\omega}\right)\exp(+i\omega t),$$

or vice-versa.

Let's try to simplify these expressions. That will be easy if we can commute the time-development operator through the annihilation and creation operators. Starting with the former,

$$\begin{aligned}\mathbf{a}\exp(-i\mathbf{H}t/\hbar) &= \mathbf{a}\exp\left[-i\omega t\left(\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right)\right]\\ &= \sum_{n=0}^{\infty}\frac{(-i\omega t)^n}{n!}\mathbf{a}\left(\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right)^n.\end{aligned}$$

Next we notice that for each term in the n^{th} power of $\left(\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right)$,

$$\begin{aligned}\mathbf{a}\left(\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right) &= \left(\mathbf{a}\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\mathbf{a}\right)\\ &= \left((\mathbf{a}^\dagger\mathbf{a}+1)\mathbf{a}+\frac{1}{2}\mathbf{a}\right)\\ &= \left(\mathbf{a}^\dagger\mathbf{a}+\frac{3}{2}\right)\mathbf{a}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{a}\exp(-i\mathbf{H}t/\hbar) &= \sum_{n=0}^{\infty}\frac{(-i\omega t)^n}{n!}\mathbf{a}\left(\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{(-i\omega t)^n}{n!}\left(\mathbf{a}^\dagger\mathbf{a}+\frac{3}{2}\right)^n\mathbf{a}\\ &= \exp\left[-i\omega t\left(\mathbf{a}^\dagger\mathbf{a}+\frac{3}{2}\right)\right]\mathbf{a}\\ &= \exp\left[-i\omega t\left(\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right)\right]\exp(-i\omega t)\,\mathbf{a}.\end{aligned}$$

Likewise,

$$\begin{aligned}\mathbf{a}^\dagger\left(\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right) &= \left(\mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right)\\ &= \left(\mathbf{a}^\dagger(\mathbf{a}\mathbf{a}^\dagger-1)+\frac{1}{2}\right)\\ &= \left(\mathbf{a}^\dagger\mathbf{a}-\frac{1}{2}\right)\mathbf{a}^\dagger,\end{aligned}$$

so

$$\begin{aligned}\mathbf{a}^\dagger\exp(-i\mathbf{H}t/\hbar) &= \sum_{n=0}^{\infty}\frac{(-i\omega t)^n}{n!}\mathbf{a}^\dagger\left(\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{(-i\omega t)^n}{n!}\left(\mathbf{a}^\dagger\mathbf{a}-\frac{1}{2}\right)^n\mathbf{a}^\dagger\\ &= \exp\left[-i\omega t\left(\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right)\right]\exp(+i\omega t)\,\mathbf{a}^\dagger.\end{aligned}$$

We thus have

$$\sqrt{\frac{\hbar}{2m\omega}}\langle\Psi(0)\,|\exp(+i\mathbf{H}t/\hbar)\,\mathbf{a}\exp(-i\mathbf{H}t/\hbar)\,|\Psi(0)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\exp(-i\omega t)\langle\Psi(0)|\mathbf{a}|\Psi(0)\rangle,$$

$$\sqrt{\frac{\hbar}{2m\omega}}\langle\Psi(0)\,|\exp(+i\mathbf{H}t/\hbar)\,\mathbf{a}^\dagger\exp(-i\mathbf{H}t/\hbar)\,|\Psi(0)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\exp(+i\omega t)\left\langle\Psi(0)|\mathbf{a}^\dagger|\Psi(0)\right\rangle.$$

But now we note that

$$\sqrt{\frac{\hbar}{2m\omega}}\langle\Psi(0)|\mathbf{a}|\Psi(0)\rangle = \frac{1}{2}\left(\langle\mathbf{x}\rangle_0 + i\frac{\langle\mathbf{p}\rangle_0}{m\omega}\right),$$

$$\sqrt{\frac{\hbar}{2m\omega}}\left\langle\Psi(0)|\mathbf{a}^\dagger|\Psi(0)\right\rangle = \frac{1}{2}\left(\langle\mathbf{x}\rangle_0 - i\frac{\langle\mathbf{p}\rangle_0}{m\omega}\right).$$

Hence, the classical dynamical equation for first-order moments

$$\begin{aligned}\langle\mathbf{x}\rangle_t &= \langle\mathbf{x}\rangle_0\cos(\omega t) + \frac{\langle\mathbf{p}\rangle_0}{m\omega}\sin(\omega t)\\ &= \frac{1}{2}\left(\langle\mathbf{x}\rangle_0 - i\frac{\langle\mathbf{p}\rangle_0}{m\omega}\right)\exp(+i\omega t) + \frac{1}{2}\left(\langle\mathbf{x}\rangle_0 + i\frac{\langle\mathbf{p}\rangle_0}{m\omega}\right)\exp(-i\omega t).\end{aligned}$$

is automatically satisfied for all initial states of the LHO! As we have shown, this is guarenteed by the form of the Hamiltonian operator.

It turns out that our second requirement is a bit more restrictive. We would like

$$\begin{aligned}(\Delta\mathbf{x})^2 &= \langle\mathbf{x}^2\rangle - \langle\mathbf{x}\rangle^2\\ &\propto \left\langle\left(\mathbf{a}+\mathbf{a}^\dagger\right)^2\right\rangle - \left\langle\mathbf{a}+\mathbf{a}^\dagger\right\rangle^2\\ &= \left\langle\mathbf{a}^2 + \mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a} + \left(\mathbf{a}^\dagger\right)^2\right\rangle - \left(\langle\mathbf{a}\rangle + \left\langle\mathbf{a}^\dagger\right\rangle\right)^2\\ &= 1 + \left\langle\mathbf{a}^2 + 2\mathbf{a}^\dagger\mathbf{a} + \left(\mathbf{a}^\dagger\right)^2\right\rangle - \left(\langle\mathbf{a}\rangle + \left\langle\mathbf{a}^\dagger\right\rangle\right)^2\end{aligned}$$

to remain small for all times. A powerful way to achieve this is by deriving *eigenstates* of the annihilation operator,

$$\mathbf{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

Such states are known as *coherent states* of the harmonic oscillator. They clearly satisfy

$$\langle\mathbf{a}\rangle = \alpha,$$

$$\left\langle\mathbf{a}^\dagger\right\rangle = \alpha^*.$$

Note that the eigenvalues α are allowed to be complex because the annihilation operator is not Hermitian. If we prepare the initial state of the harmonic oscillator to be $|\alpha\rangle$, then clearly at $t = 0$

$$\begin{aligned}(\Delta\mathbf{x})^2 &\propto 1 + \left\langle\mathbf{a}^2 + 2\mathbf{a}^\dagger\mathbf{a} + \left(\mathbf{a}^\dagger\right)^2\right\rangle - \left(\langle\mathbf{a}\rangle + \left\langle\mathbf{a}^\dagger\right\rangle\right)^2\\ &= 1 + \alpha^2 + 2|\alpha|^2 + (\alpha^*)^2 - (\alpha+\alpha^*)^2\\ &= 1.\end{aligned}$$

But how will this uncertainty evolve with time? To answer this we must derive some general properties of coherent states.

The ground state $|0\rangle$ of the LHO is a coherent state with $\alpha = 0$. In much the same way that we generated further eigenstates of the number operator by applying the creation operator, we can generate new coherent states by acting on $|0\rangle$ with a displacement operator $\mathbf{D}(\alpha)$:

$$|\alpha\rangle = \mathbf{D}(\alpha)|0\rangle,$$
$$\mathbf{D}(\alpha) = \exp\left(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a}\right).$$

Just as we found we could derive many properties of number states by working with the annihilation and creation operators, we can derive many important properties of coherent states by working with the displacement operator.

First of all we note that $\mathbf{D}(\alpha)$ is unitary, so $|\alpha\rangle$ as defined above is automatically normalized:

$$\begin{aligned}\langle\alpha|\alpha\rangle &= \left\langle 0|\mathbf{D}^\dagger(\alpha)\mathbf{D}(\alpha)|0\right\rangle \\ &= \langle 0|0\rangle \\ &= 1.\end{aligned}$$

Next we need to see how it commutes with the time-development operator. Towards this end, we must first derive some intermediate results using the general formula (Merzbacher 3.59)

$$\exp(\lambda\mathbf{A})\mathbf{B}\exp(-\lambda\mathbf{A}) = \exp(\lambda\gamma)\mathbf{B},$$

which is valid when $[\mathbf{A},\mathbf{B}] = \gamma\mathbf{B}$ (λ and γ must be constants). If we define

$$\mathbf{R}(\varphi) = \exp\left(i\varphi\mathbf{a}^\dagger\mathbf{a}\right),$$

we see that

$$\mathbf{R}^\dagger(\varphi)\,\mathbf{a}\,\mathbf{R}(\varphi) = \exp(i\varphi)\,\mathbf{a}$$
$$\mathbf{R}^\dagger(\varphi)\,\mathbf{a}^\dagger\,\mathbf{R}(\varphi) = \exp(-i\varphi)\,\mathbf{a}^\dagger$$

since

$$\begin{aligned}\left[\mathbf{a}^\dagger\mathbf{a},\mathbf{a}\right] &= \mathbf{a}^\dagger\mathbf{a}\mathbf{a} - \mathbf{a}\mathbf{a}^\dagger\mathbf{a} \\ &= \mathbf{a}^\dagger\mathbf{a}\mathbf{a} - \left(\mathbf{a}^\dagger\mathbf{a}+1\right)\mathbf{a} \\ &= -\mathbf{a}, \quad (\gamma = -1) \\ \left[\mathbf{a}^\dagger\mathbf{a},\mathbf{a}^\dagger\right] &= \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a} \\ &= \mathbf{a}^\dagger\left(\mathbf{a}^\dagger\mathbf{a}+1\right) - \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a} \\ &= \mathbf{a}^\dagger. \quad (\gamma = +1)\end{aligned}$$

From the above we see that

$$\mathbf{R}(\varphi)\mathbf{a} = \exp(-i\varphi)\mathbf{a}\mathbf{R}(\varphi),$$
$$\mathbf{R}(\varphi)\mathbf{a}^\dagger = \exp(i\varphi)\mathbf{a}^\dagger\mathbf{R}(\varphi).$$

Now we can compute

$$\begin{aligned}\mathbf{R}(\varphi)\mathbf{D}(\alpha) &= \mathbf{R}(\varphi)\exp\left(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a}\right) \\ &= \mathbf{R}(\varphi)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\alpha e^{i\varphi}\mathbf{a}^\dagger - \alpha^* e^{-i\varphi}\mathbf{a}\right)^n\mathbf{R}(\varphi) \\ &= \mathbf{D}(\alpha e^{i\varphi})\mathbf{R}(\varphi).\end{aligned}$$

And since the time-development operator is simply

$$\mathbf{T}(t,0) = \exp\left(-i\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)t\right)$$
$$= \exp\left(-\frac{i}{2}\omega t\right)\mathbf{R}(-\omega t),$$

we have

$$\mathbf{T}(t,0)\mathbf{D}(\alpha) = \exp\left(-\frac{i}{2}\omega t\right)\mathbf{D}(\alpha e^{-i\omega t})\exp\left(-i\omega t\,\mathbf{a}^\dagger\mathbf{a}\right).$$

Using this nice result, we are finally able to see exactly how coherent states evolve in time:

$$\mathbf{T}(t,0)|\alpha\rangle = \mathbf{T}(t,0)\,\mathbf{D}(\alpha)|0\rangle$$
$$= \exp\left(-\frac{i}{2}\omega t\right)\mathbf{D}(\alpha e^{-i\omega t})\exp\left(-i\omega t\,\mathbf{a}^\dagger\mathbf{a}\right)|0\rangle$$
$$= \exp\left(-\frac{i}{2}\omega t\right)|\alpha e^{-i\omega t}\rangle.$$

Amazingly, coherent states evolve into other coherent states whose phases are different by $-\omega t$ (plus an overall phase factor). Getting back to our issue of classical behavior, we thus see that $(\Delta\mathbf{x})^2 \propto 1$ for *all* times when the oscillator is initially prepared in a coherent state. Not bad!

So what do these things actually look like? We can solve for the coherent state wave function by solving the eigenvalue equation in the position representation,

$$\mathbf{a}|\alpha\rangle = \alpha|\alpha\rangle$$
$$\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{2m\omega}\frac{d}{dx}\right)\psi_\alpha(x) = \alpha\psi_\alpha(x).$$

The solutions are easily seen to be displaced Gaussians,

$$\psi_\alpha(x) = C'\exp\left[-\frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\,\alpha\right)^2\right].$$

Hence the name ‘displacement operator.’ Note that unless α is real this changes both $\langle\mathbf{x}\rangle$ and $\langle\mathbf{p}\rangle$, in accordance with

$$\alpha = \langle\mathbf{a}\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(\langle\mathbf{x}\rangle + i\frac{\langle\mathbf{p}\rangle}{m\omega}\right).$$