Ph195b, lecture notes for 2/5/02

One of the most important things to retain from basic QM is a solid appreciations of the similarities and differences between quantum and classical models of the harmonic oscillator. Today's lecture is prepared with this in mind, as we will first explore some of the elegant and improbable ways that quantum mechanics reproduces classical features of the harmonic oscillator, but then move on to consider some novel quantum features of harmonic oscillators that have been exploited for remarkable achievements in the laser cooling of atoms and ions. Please have a look at the papers referenced.

Thinking 'physically' about the LHO

According to popular opinion, the defining feature of the quantum harmonic oscillator is uniform spacing of energy levels:

$$\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

What has this got to do with intuitive 'physical' properties of a harmonic oscillator? It's quite instructive to see how quantum mechanics draws the connection.

Basically, the crucial point is that a harmonic oscillator's Bohr frequencies are all multiples of ω. Hence the time evolution of all operator moments are either constant or periodic in ω^{-1}:

$$\begin{aligned}\langle \mathbf{O} \rangle &= \sum_{n,k=0}^{\infty} c_n^* c_k \langle n | \mathbf{O} | k \rangle \exp(-i(k-n)\omega t) \\ &= \sum_{m=-\infty}^{\infty} C_m \exp(-im\omega t) \\ &= \sum_{m=0}^{\infty} R_m \cos(m\omega t) \quad \left(\text{assuming } \mathbf{O} = \mathbf{O}^\dagger\right).\end{aligned}$$

For certain operators the lowest frequency in this Fourier series might be not ω but some multiple of ω. For example,

$$\frac{\hbar\omega}{4}\left[\mathbf{a}^2 + \left(\mathbf{a}^\dagger\right)^2\right] = \left(\frac{1}{2} m\omega^2 \mathbf{x}^2 - \frac{\mathbf{p}^2}{2m}\right)$$

only has non-zero matrix elements for $|n-k| = 2$, so it is periodic in $\frac{1}{2}\omega^{-1}$. This makes perfect sense if you think about the classical analog, having to do with the *parity* symmetry of the LHO Hamiltonian.

The expectation values of position and momentum, however, are constrained to be periodic in ω^{-1} because of the forms of the corresponding operators. Since

$$\mathbf{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\mathbf{a} + \mathbf{a}^\dagger\right)$$

for example, there exists a selection rule

$$\langle n|\mathbf{x}|k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\delta_{k,n-1} + \sqrt{n+1}\,\delta_{k,n+1}\right],$$

from which follows

$$\langle \mathbf{x}\rangle_t = \sqrt{\frac{2\hbar}{m\omega}}\sum_{n=1}^{\infty}\sqrt{n}\,|c_n||c_{n-1}|\cos(\omega t + \varphi_{n-1} - \varphi_n).$$

That is, no frequencies higher than ω can appear in the evolution of $\langle \mathbf{x}\rangle$, and the only way for $\langle \mathbf{x}\rangle$ to be constant is if there are no non-zero 'adjacent' coefficients in the energy basis. In fact, we showed that the mean value of position evolves according to the classical equation

$$\langle \mathbf{x}\rangle_t = \langle \mathbf{x}\rangle_0 \cos\omega t + \langle \mathbf{p}\rangle_0 \sin\omega t$$

for *all* initial states.

Hence we see that uniform spacing of energy levels translates directly into harmonic variation of the expectation values of observables.

It is also often said that the zero-point energy $\varepsilon_0 = \frac{1}{2}\hbar\omega$ is a specifically 'quantum' feature of the quantum harmonic oscillator. Why is this so?

We have already noted that all energy eigenstates of the LHO satisfy $\langle \mathbf{x}\rangle = \langle \mathbf{p}\rangle = 0$, and that $\langle \mathbf{H}\rangle = 0$ is essentially forbidden by the Uncertainty Principle. In particular,

$$\begin{aligned}
\langle \mathbf{H}\rangle &= \frac{1}{2}m\omega^2\langle \mathbf{x}^2\rangle + \frac{\langle \mathbf{p}^2\rangle}{2m} \\
&= \frac{1}{2}m\omega^2(\Delta\mathbf{x})^2 + \frac{(\Delta\mathbf{p})^2}{2m} \\
&\geq \frac{1}{2}m\omega^2(\Delta\mathbf{x})^2 + \frac{\hbar^2}{8m(\Delta\mathbf{x})^2}
\end{aligned}$$

which cannot be zero for finite m and ω since $\Delta\mathbf{x}\Delta\mathbf{p} \geq \hbar/2$. If we minimize the above expression with respect to $\Delta\mathbf{x}$, we find

$$\begin{aligned}
0 &= \frac{\partial}{\partial(\Delta\mathbf{x})}\left(\frac{1}{2}m\omega^2(\Delta\mathbf{x})^2 + \frac{\hbar^2}{8m(\Delta\mathbf{x})^2}\right) \\
&= m\omega^2(\Delta\mathbf{x}) - \frac{\hbar^2}{4m}(\Delta\mathbf{x})^{-3}, \\
(\Delta\mathbf{x})^4 &= \frac{\hbar^2}{4m^2\omega^2}, \\
\Delta\mathbf{x} &= \sqrt{\frac{\hbar}{2m\omega}} \quad \text{(minimum energy)}.
\end{aligned}$$

Looking at the expression for the ground-state wave function

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right),$$

we see that this is indeed true:

$$
\begin{aligned}
\Delta\mathbf{x} &= \left[\left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{m\omega}{\hbar} x^2 \right) \right]^{1/2} \\
&= \left[\left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{\hbar}{2m\omega} \sqrt{\frac{\pi\hbar}{m\omega}} \right]^{1/2} \\
&= \sqrt{\frac{\hbar}{2m\omega}} .
\end{aligned}
$$

Interestingly enough, making $\Delta\mathbf{x}$ *either* smaller *or* larger than this characteristic value (recall our dimensional analysis) would increase the energy of the state.

Hence, the value of the zero point energy follows from the form of the LHO Hamiltonian plus the exact quantum mechanical commutator of **x** and **p**.

And why is the ground state Gaussian? Best I can tell, Gaussians are the prefered function for harmonic oscillators because of the differential form of the annihilation operator,

$$\mathbf{a} \rightarrow \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{\hbar}{2m\omega} \frac{\partial}{\partial x} \right).$$

Coherent states and the Wigner function

We saw last time that the solutions of the eigenvalue equation

$$\mathbf{a}\psi(x) = \alpha\psi(x)$$

(where α is allowed to be complex) are in fact *all* displaced Gaussians,

$$\psi_\alpha(x) = C' \exp\left[-\frac{m\omega}{2\hbar} \left(x - \sqrt{\frac{2\hbar}{m\omega}}\, \alpha \right)^2 \right].$$

All such ‘coherent states’ $|\alpha\rangle$ can be generated from the ground state by action of a displacement operator

$$\mathbf{D}(\alpha) = \exp\left(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a} \right),$$

and have constant uncertainties

$$
\begin{aligned}
\Delta\mathbf{x} &= \sqrt{\frac{\hbar}{2m\omega}} \\
\Delta\mathbf{p} &= \frac{\hbar}{2}(\Delta\mathbf{x})^{-1} \\
&= \sqrt{\frac{m\hbar\omega}{2}} .
\end{aligned}
$$

We previously noted that coherent states evolve quite simply under the LHO Hamiltonian,

$$\mathbf{T}(t,0)|\alpha\rangle = \exp\left(-\frac{i}{2}\omega t \right) |\alpha e^{-i\omega t}\rangle.$$

Since

$$\alpha = \langle \mathbf{a} \rangle = \sqrt{\frac{m\omega}{2\hbar}} \left(\langle \mathbf{x} \rangle + i \frac{\langle \mathbf{p} \rangle}{m\omega} \right),$$

we see that coherent states evolve according to the classical relation between x and p, as they must.

Looking at these expressions, you get the idea that it might be neat to represent the evolution of coherent states on the complex plane. That is, on a graph with axes corresponding to

$$\mathrm{Re}\,\alpha = \sqrt{\frac{m\omega}{2\hbar}}\langle \mathbf{x} \rangle,$$

$$\mathrm{Im}\,\alpha = \sqrt{\frac{1}{2m\hbar\omega}}\langle \mathbf{p} \rangle,$$

we see that any given coherent state is represented by a point and that its evolution traces out a circle of radius $|\alpha|$. In fact, since we know that coherent states are Gaussians, we could represent their position and momentum uncertainties on the same graph by drawing a circle of radius $\frac{1}{2}$. Two things to note about this representation are that $\langle \mathbf{H} \rangle = \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right)$ is monotonically related to distance from the origin, and that the 'phase' uncertainty $\Delta\left(\tan^{-1}\left[\frac{\mathrm{Im}\alpha}{\mathrm{Re}\alpha}\right]\right)$ is seen to decrease as $|\alpha|$ gets larger.

In fact, this sort of representation can be made rigorous and corresponds to the famous Wigner representation,

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dx' \, \exp(-ipx'/\hbar)\psi^*\left(x - \frac{x'}{2}\right)\psi\left(x + \frac{x'}{2}\right).$$

Here $W(x,p)$ is a function over the x,p plane (equivalently our α plane) that may loosely be thought of as a joint probability distribution over x and $p,$ in the sense that

$$\int_{-\infty}^{+\infty} dp\, W(x,p) = |\psi(x)|^2,$$

$$\int_{-\infty}^{+\infty} dx\, W(x,p) = |\bar{\psi}(p)|^2.$$

Be careful to note that $W(x,p)$ is only a *quasi*-probability distribution, since it can be negative! For coherent states, the Wigner function is simply a $2D$ Gaussian with widths $\Delta\mathbf{x}$ and $\Delta\mathbf{p},$ centered at the point $(\langle \mathbf{x} \rangle, \langle \mathbf{p} \rangle)$.

Squeezed states and squeezing operators

The ground state of the LHO is a stationary Gaussian centered at $x = 0,$ with width $\Delta\mathbf{x} = \sqrt{\hbar/2m\omega}$. As shown above, any state with narrower width must have higher energy than the ground state. However, a narrow state cannot correspond to any energy eigenstate, since they all satisfy

$$\begin{aligned}(\Delta\mathbf{x})_n &= \sqrt{\langle n|\mathbf{x}^2|n\rangle} \\ &= \sqrt{\frac{\hbar}{2m\omega}\left\langle \mathbf{a}^2 + \mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a} + \left(\mathbf{a}^\dagger\right)^2\right\rangle} \\ &= \sqrt{\frac{\hbar}{2m\omega}\left\langle \mathbf{a}^2 + 2\mathbf{a}^\dagger\mathbf{a} + 1 + \left(\mathbf{a}^\dagger\right)^2\right\rangle} \\ &= \sqrt{\frac{\hbar}{2m\omega}(2n+1)}.\end{aligned}$$

Therefore, a state with $\Delta\mathbf{x} < \sqrt{\hbar/2m\omega}$ is cannot be stationary. Since the evolution of $\Delta\mathbf{x}$ must be either constant or stationary for a harmonic oscillator, we infer that the width of an initially-narrow state must follow a regular 'breathing' evolution.

Let's think back for a moment to the Gaussian ground-state wave function and ponder its stationarity. We noted above that its width and momentum spread stand in particularly quantum-mechanical relation, with $\Delta\mathbf{x}\Delta\mathbf{p} = \hbar/2$. Of course, we can also relate position to momentum for a classical harmonic oscillator:

$$\begin{aligned} \frac{1}{2}m\omega^2(\Delta\mathbf{x})^2 &= \frac{p^2}{2m}, \\ p &= \sqrt{\frac{\hbar\omega}{4}(2m)} \\ &= \sqrt{\frac{m\hbar\omega}{2}} \\ &= \Delta\mathbf{p}. \end{aligned}$$

If we momentarily put aside what we know about the subtleties of interpreting the wave function, we can imagine some analogy with a *probabilistic* distribution of classical particles in a harmonic potential well. Such a distribution should have similarly related position and momentum widths if it is to be stationary. Hence, it seems reasonable that this sort of 'balance' between $\Delta\mathbf{x}$ and $\Delta\mathbf{p}$ might be necessary for quantum-mechanical stationarity. Indeed, for the higher eigenstates we have

$$\frac{1}{2}m\omega^2(\Delta\mathbf{x})^2 = \frac{\hbar\omega}{4}(2n+1)$$

from the result derived above, and similarly

$$\begin{aligned} \frac{(\Delta\mathbf{p})^2}{2m} &= \frac{\hbar\omega}{4}\left\langle -\mathbf{a}^2 + \mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a} - \left(\mathbf{a}^\dagger\right)^2\right\rangle \\ &= \frac{\hbar\omega}{4}(2n+1). \end{aligned}$$

This of course recovers what we know about the energy eigenvalues (the sum of these two quantities). It is interesting to note that higher-n states satisfy the natural 'classical' relation between $\Delta\mathbf{x}$ and $\Delta\mathbf{p}$, but not the 'quantum' one since

$$\Delta\mathbf{x}\Delta\mathbf{p} = \frac{\hbar}{2}(2n+1).$$

In this regard, coherent states do a much better job of being quantum and classical at the same time. Anyway, getting back to our original train of thought... For any state with $\Delta\mathbf{x} < \sqrt{\hbar/2m\omega}$ and $\Delta\mathbf{p}$ correspondingly greater than $\sqrt{m\hbar\omega/2}$, we can imagine that our imaginary distribution of particles is out of balance and therefore must expand.

One particularly simple class of narrow states to consider is the *squeezed* states. Formally, these may be thought of as another class of states that can be generated from the ground state by the action of a unitary operator, this time the squeezing operator

$$\mathbf{S}(\xi) = \exp\left(\frac{\xi}{2}\left[\mathbf{a}^2 - \left(\mathbf{a}^\dagger\right)^2\right]\right).$$

Here ξ is a parameter associated with the squeezed state $\mathbf{S}(\xi)|0\rangle$, in much the same way as α is associated with a coherent state. However, a much better way to think about squeezed states is as follows.

Suppose we start with a harmonic potential with oscillation frequency ω', prepare the ground state of this potential, and then suddenly switch to a harmonic potential with oscillation frequency $\omega < \omega'$. If we make the switch sufficiently quickly, we will have made a squeezed state of the ω potential, with initial width $\sqrt{\hbar/2m\omega'} < \sqrt{\hbar/2m\omega}$. To see this quantitatively, note that we can define annihilation and creation operators for both the initial $(\mathbf{b}, \mathbf{b}^\dagger)$ and final $(\mathbf{a}, \mathbf{a}^\dagger)$ Hamiltonians in terms of the same position and momentum operators (it's the same particle, after all):

$$\mathbf{b} = \sqrt{\frac{m\omega'}{2\hbar}}\left(\mathbf{x} + i\frac{\mathbf{p}}{m\omega'}\right),$$
$$\mathbf{b}^\dagger = \sqrt{\frac{m\omega'}{2\hbar}}\left(\mathbf{x} - i\frac{\mathbf{p}}{m\omega'}\right),$$
$$\mathbf{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\mathbf{x} + i\frac{\mathbf{p}}{m\omega}\right),$$
$$\mathbf{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\mathbf{x} - i\frac{\mathbf{p}}{m\omega}\right).$$

Since $\mathbf{x}$ is $\mathbf{x}$ and $\mathbf{p}$ is $\mathbf{p}$, we can clearly solve for one set in terms of the other:

$$\begin{aligned}\mathbf{b} &= \frac{1}{2}\left(\sqrt{\omega'/\omega} + \sqrt{\omega/\omega'}\right)\mathbf{a} + \frac{1}{2}\left(\sqrt{\omega'/\omega} - \sqrt{\omega/\omega'}\right)\mathbf{a}^\dagger \\ &= \lambda\mathbf{a} + \nu\mathbf{a}^\dagger, \\ \mathbf{b}^\dagger &= \frac{1}{2}\left(\sqrt{\omega'/\omega} + \sqrt{\omega/\omega'}\right)\mathbf{a}^\dagger + \frac{1}{2}\left(\sqrt{\omega'/\omega} - \sqrt{\omega/\omega'}\right)\mathbf{a} \\ &= \lambda\mathbf{a}^\dagger + \nu\mathbf{a}.\end{aligned}$$

Note that $\lambda^2 - \nu^2 = 1$, and that $\lambda > 1$. The state we prepare by the above procedure will (initially) satisfy

$$\mathbf{b}|0'\rangle = 0.$$

(This is a 'squeezed vacuum' state; other squeezed states exist with $\mathbf{b}|\beta\rangle = \beta$.) We thus know that

$$\Delta\mathbf{x} = \sqrt{\frac{\hbar}{2m\omega'}} = \sqrt{\frac{\hbar}{2m\omega}}(\lambda - \nu),$$
$$\Delta\mathbf{p} = \sqrt{\frac{m\hbar\omega'}{2}} = \sqrt{\frac{m\hbar\omega}{2}}(\lambda + \nu),$$

hence the balance between position and momentum widths is shifted towards narrow $\Delta\mathbf{x}$. We can immediately see that $|0'\rangle$ is not a stationary state, as the eigenvalue equation

$$\mathbf{b}|0'\rangle = \left(\lambda\mathbf{a} + \nu\mathbf{a}^\dagger\right)|0'\rangle = 0$$

will generate a recursion relation between the coefficients in an expansion

$$\left(\lambda\mathbf{a} + \nu\mathbf{a}^\dagger\right)|0'\rangle = \left(\lambda\mathbf{a} + \nu\mathbf{a}^\dagger\right)\sum_{n=0}^{\infty} c_n|n\rangle = 0,$$

$$\lambda\sum_{n=0}^{\infty}\sqrt{n+1}\,c_{n+1}|n\rangle + \nu\sum_{n=1}^{\infty}\sqrt{n}\,c_{n-1}|n\rangle = 0$$

from which we have

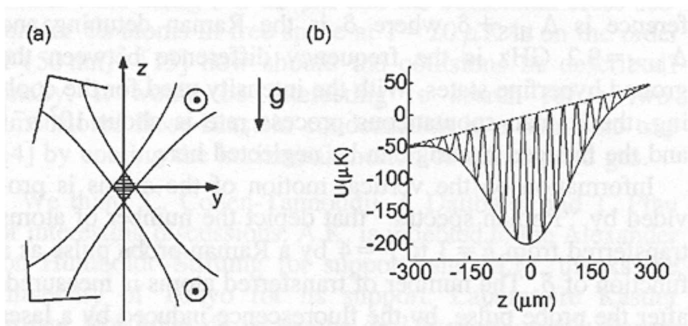
$$c_1 = 0,$$
$$c_{n+1} = \frac{-\nu\sqrt{n}}{\lambda\sqrt{n+1}}c_{n-1}, \quad (n \geq 1).$$

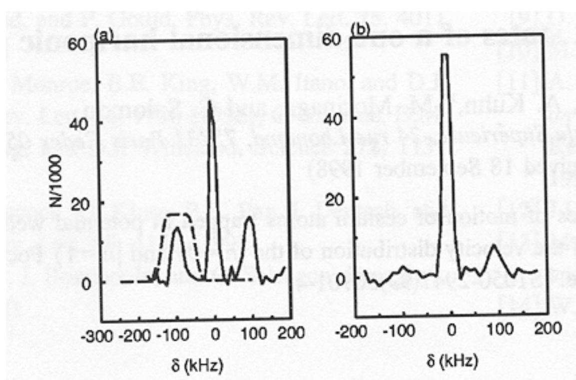
This clearly converges, since $\lambda > \nu$, and involves only states of even n.

Quantum harmonic oscillators in atomic physics

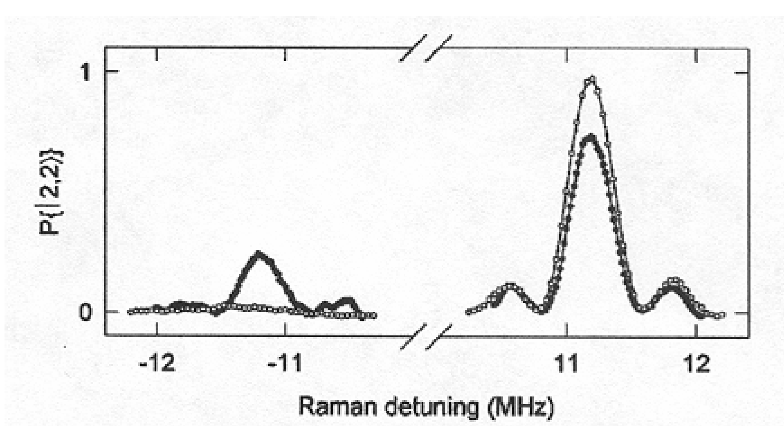
1. Ions in rf Paul traps, neutral atoms in dipole-force traps
2. Resolved-sideband cooling to the ground state
3. Preparation of coherent, squeezed, and number states
4. Reconstruction of the Wigner distribution
5. Quantum optical realizations; coherent and squeezed states of light

I. Bouchoule *et al*, Phys. Rev. A **59**, R8 (1999).





C. Monroe *et al*, Phys. Rev. Lett. **75**, 4011 (1995).



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