Ph195b lecture notes for 2/25/02

Angular momentum and rotations

Consider a geometrical transformation R that corresponds to a rotation of Euclidean space, such that

$$\vec{r}' = R\vec{r}.$$

We know that such rotations can be represented on the vectors $\vec{r}$ themselves via rotation matrices, *e.g.*,

$$R_x(\theta) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

for a rotation by angle θ about the x axis. Similarly,

$$R_y(\theta) \leftrightarrow \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad R_z(\theta) \leftrightarrow \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is important to note that rotation matrices corresponding to different axes do not commute in general:

$$R_x\left(\frac{\pi}{2}\right)R_z\left(\frac{\pi}{2}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$R_z\left(\frac{\pi}{2}\right)R_x\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

This of course reflects the fact that rotation operations don't commute! (Try it with a solid object.)

The general commutation structure of the rotation matrices can in fact be summarized by commutation relations between three generators. To see what these algebraic generators are, note that

$$
\begin{aligned}
R_x(\theta) &\leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i\theta) & 0 \\ 0 & 0 & \exp(-i\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\theta & 0 \\ 0 & 0 & -i\theta \end{pmatrix}^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\theta & 0 \\ 0 & 0 & -i\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \right]^n,
\end{aligned}
$$

where the last step could be taken because the eigenvector matrices are unitary. Continuing, we have

$$
\begin{aligned}
R_x(\theta) &\leftrightarrow \exp\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\theta & 0 \\ 0 & 0 & -i\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \right] \\
&= \exp\left[i\theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \right] \\
&= \exp\left[i\theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \right] \equiv \exp(i\theta L_x).
\end{aligned}
$$

Hence we have a (Hermitian) generator for rotations about $\hat{x}$. Similarly, one can derive

$$
R_y(\theta) = \exp\left[i\theta \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \right], \quad R_z(\theta) = \exp\left[i\theta \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right].
$$

Hence all rotation matrices can be written in terms of algebraic sums (Taylor expansions) involving only three distinct “basis” matrices L_x, L_y, L_z whose commutators can easily be computed:

$$[L_x, L_y] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -iL_z,$$

$$[L_y, L_z] = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -iL_x,$$

$$[L_z, L_x] = -iL_y.$$

We finally recognize a set of commutation relations that look (up to a minus sign?) very much like those of the angular momentum algebra we have considered for quantum systems!

Under Euclidean rotations, wave functions are mapped according to

$$\psi'(\vec{r}\,') = \psi(\vec{r}),$$
$$\psi'(R\vec{r}) = \psi(\vec{r}),$$
$$\psi'(\vec{r}) = \psi\left(R^{-1}\vec{r}\right).$$

We would like to define a Hilbert-space operator **R** such that

$$|\Psi'\rangle = \mathbf{R}|\Psi\rangle,$$

and therefore

$$\psi'(\vec{r}) = \langle\vec{r}|\Psi'\rangle = \langle\vec{r}|\mathbf{R}|\Psi\rangle,$$
$$= \left\langle R^{-1}\vec{r}\,|\Psi\right\rangle,$$

where the last equation follows from $\psi'(\vec{r}) = \psi\left(R^{-1}\vec{r}\right)$.

So defined, the rotation operator **R** is *linear*

$$\langle\vec{r}|\mathbf{R}(\lambda_1|\Psi_1\rangle + \lambda_2|\Psi_2\rangle) = \left\langle R^{-1}\vec{r}\,\right|(\lambda_1|\Psi_1\rangle + \lambda_2|\Psi_2\rangle)$$
$$= \lambda_1\left\langle R^{-1}\vec{r}\,|\Psi_1\right\rangle + \lambda_2\left\langle R^{-1}\vec{r}\,|\Psi_2\right\rangle$$
$$= \lambda_1\langle\vec{r}|\mathbf{R}|\Psi_1\rangle + \lambda_2\langle\vec{r}|\mathbf{R}|\Psi_2\rangle.$$

From the evident property

$$\langle\vec{r}|\mathbf{R} = \left\langle R^{-1}\vec{r}\,\right|,$$

(since we decided $\langle\vec{r}|\mathbf{R}|\Psi\rangle = \left\langle R^{-1}\vec{r}\,|\Psi\right\rangle$ for all states) we also have

$$\mathbf{R}^{\dagger}|\vec{r}\rangle = \left|R^{-1}\vec{r}\right\rangle.$$

Combining this with the basic property

$$\mathbf{R}|\vec{r}\rangle = |R\vec{r}\rangle,$$

we see that

$$\mathbf{R}\mathbf{R}^{\dagger} = \mathbf{R}^{\dagger}\mathbf{R} = \mathbf{1},$$

so **R** is *unitary*.

Note also that the mapping between R and **R** preserves the *group* properties of rotations (the product of two rotations is still a rotation, there is an identity element in the group – rotation by zero angle about any axis, and every rotation has an inverse) in the same way as our mapping between rotations and rotation matrices earlier in this lecture. In particular, if

$$R_1 R_2 = R_3,$$

then

$$\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}_3.$$

The set of operators **R** is therefore said to constitute a *representation* of the rotation group, on the corresponding Hilbert space (as opposed to the vector space R^3). It is important to remember that two rotations do not commute $(R_1R_2 \neq R_2R_1)$ except if they act about the same axis.

Infinitesimal rotations:

$$R_{\mathbf{u}}(d\alpha)\vec{r} = \vec{r} + d\alpha\,(\hat{u}\times\vec{r}).$$

In particular, for $\hat{u} = \hat{z}$ we have

$$\begin{aligned}\psi'(\vec{r}) &= \psi\left[R_{\mathbf{z}}^{-1}(d\alpha)\vec{r}\right]\\ &= \psi[R_{\mathbf{z}}(-d\alpha)\vec{r}]\\ &= \psi[\vec{r} - d\alpha\,(\hat{z}\times\vec{r})].\end{aligned}$$

Representing the Cartesian coordinates individually,

$$\begin{aligned}\psi'(x,y,z) &= \psi(x + y\,d\alpha, y - x\,d\alpha, z)\\ &= \psi(x,y,z) - d\alpha\left[x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x}\right].\end{aligned}$$

Hence,

$$\begin{aligned}\psi'(\vec{r}) = \langle\vec{r}|\psi'\rangle &= \left\langle\vec{r}\right|\left(1 - \frac{i}{\hbar}d\alpha\mathbf{L}_z\right)|\psi\rangle\\ &= \langle\vec{r}|\mathbf{R}_{\mathbf{z}}(d\alpha)|\psi\rangle\end{aligned}$$

and we have the operator association

$$\mathbf{R}_{\mathbf{z}}(d\alpha) = \left(1 - \frac{i}{\hbar}d\alpha\mathbf{L}_z\right).$$

For a general infinitesimal rotation,

$$\mathbf{R}_{\mathbf{u}}(d\alpha) = \left(1 - \frac{i}{\hbar}d\alpha\vec{\mathbf{L}}\cdot\hat{u}\right).$$

Noting the general relation

$$\mathbf{R}_{\mathbf{u}}(\alpha + d\alpha) = \mathbf{R}_{\mathbf{u}}(\alpha)\,\mathbf{R}_{\mathbf{u}}(d\alpha)$$

(since the axis of rotation is fixed), we have

$$\mathbf{R_u}(\alpha + d\alpha) = \mathbf{R_u}(\alpha)\left(1 - \frac{i}{\hbar}d\alpha\vec{\mathbf{L}}\cdot\vec{u}\right),$$

$$\mathbf{R_u}(\alpha + d\alpha) - \mathbf{R_u}(\alpha) = \left(-\frac{i}{\hbar}d\alpha\vec{\mathbf{L}}\cdot\vec{u}\right)\mathbf{R_u}(\alpha),$$

which may formally be integrated (from $\mathbf{R_u}(0) = \mathbf{1}$) to yield

$$\mathbf{R_u}(\alpha) = \exp\left(-\frac{i}{\hbar}\alpha\vec{\mathbf{L}}\cdot\vec{u}\right).$$

Hence we see that orbital angular momentum operators appear as the generators of infinitesimal rotations, in the same way that linear momentum operators generate infinitesimal displacements.

The eigenvalue problem for $\mathbf{L}_z$ and $\mathbf{L}^2$

We first note the usual relations between Cartesian and spherical coordinates,

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta,$$

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}\left[\frac{\sqrt{x^2 + y^2}}{z}\right], \quad \varphi = \tan^{-1}\frac{y}{x}.$$

From these we have

$$\begin{aligned}
\mathbf{L}_x &= \mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y \\
&= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\
&= -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right), \\
\mathbf{L}_y &= -i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right), \\
\mathbf{L}_z &= -i\hbar\frac{\partial}{\partial\varphi},
\end{aligned}$$

and thus

$$\begin{aligned}
\mathbf{L}^2 &= \mathbf{L}_x^2 + \mathbf{L}_y^2 + \mathbf{L}_z^2 \\
&= -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right], \\
\mathbf{L}_\pm &= \pm\hbar\exp(\pm i\varphi)\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\varphi}\right).
\end{aligned}$$

The first thing to notice about these expressions is that r drops out of all the angular momentum operators. Hence the simultaneous eigenfunctions of $\mathbf{L}_z$ and $\mathbf{L}^2$ may be denoted $f(r)Y_l^m(\theta,\varphi)$, with $f(r)$ an arbitrary (well-behaved) function

$$
\begin{aligned}
\mathbf{L}_z Y_l^m(\theta,\varphi) &= -i\hbar\frac{\partial}{\partial\varphi}Y_l^m(\theta,\varphi)\\
&= m\hbar Y_l^m(\theta,\varphi),\\
\mathbf{L}^2 Y_l^m(\theta,\varphi) &= -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]Y_l^m(\theta,\varphi)\\
&= l(l+1)\hbar^2 Y_l^m(\theta,\varphi).
\end{aligned}
$$

The functions $Y_l^m(\theta,\varphi)$ that satisfy these relations are known as the *Spherical Harmonics*.

As we noted last time, a good way to start deriving eigenstates is from the condition

$$
\begin{aligned}
\mathbf{L}_+ Y_l^l(\theta,\varphi) &= \hbar\exp(i\varphi)\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\varphi}\right)Y_l^l(\theta,\varphi)\\
&= 0.
\end{aligned}
$$

From this we infer

$$
\begin{aligned}
\frac{\partial}{\partial\theta}Y_l^l(\theta,\varphi) &= -i\cot\theta\frac{\partial}{\partial\varphi}Y_l^l(\theta,\varphi),\\
\tan\theta\frac{\partial}{\partial\theta}Y_l^l(\theta,\varphi) &= \frac{1}{\hbar}\mathbf{L}_z Y_l^l(\theta,\varphi)\\
&= lY_l^l(\theta,\varphi),
\end{aligned}
$$

hence

$$
\begin{aligned}
\sin\theta\frac{\partial}{\partial\theta}Y_l^l(\theta,\varphi) &= l\cos\theta Y_l^l(\theta,\varphi),\\
Y_l^l(\theta,\varphi) &= c_l(\sin\theta)^l F_l(\varphi).
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\mathbf{L}_z Y_l^l(\theta,\varphi) &= l\hbar Y_l^l(\theta,\varphi),\\
-i\frac{\partial}{\partial\varphi}F_l(\varphi) &= lF_l(\varphi),\\
F_l(\varphi) &= \exp(il\varphi),
\end{aligned}
$$

so

$$
Y_l^l(\theta,\varphi) = c_l(\sin\theta)^l\exp(il\varphi).
$$

We can define the constant factor c_l such that the Spherical Harmonics are normalized with respect to integration over solid angle,

$$
\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta d\theta\,|Y_l^l(\theta,\varphi)|^2 = |c_l|^2\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta d\theta\,(\sin\theta)^{2l} = 1,
$$

which yields

$$
|c_l| = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}.
$$

It is customary to choose c_l real, for consistency with the analogous phase convention for the matrix elements of $\mathbf{L}_\pm$ (from last lecture). Note that by normalizing the $Y_l^l(\theta,\varphi)$ themselves, we induce a requirement that

$$
\int_0^{\infty}r^2dr|f(r)|^2 = 1
$$

for the radial component of the wave function.

Having solved the 'extremal' case $m = l$, we can now obtain Spherical Harmonics with $m \neq l$ by acting on $Y_l^l(\theta,\varphi)$ with the lowering operator. In the last lecture we derived exact matrix elements for $\mathbf{L}_-$, and we may thus write

$$\mathbf{L}_- Y_l^m(\theta,\varphi) = \hbar\sqrt{(l+m)(l-m+1)}\, Y_l^{m-1}(\theta,\varphi).$$

This can be shown to lead to the general closed expressions

$$\begin{aligned} Y_l^m(\theta,\varphi) &= \sqrt{\frac{(l+m)!}{(2l)!(l-m)!}}\left(\frac{\mathbf{L}_-}{\hbar}\right)^{l-m} Y_l^l(\theta,\varphi) \\ &= \frac{(-1)^l}{2^l l!}\sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}\exp(im\varphi)(\sin\theta)^{-m}\frac{d^{l-m}}{d(\cos\theta)^{l-m}}(\sin\theta)^{2l}. \end{aligned}$$

It is worth noting that a distinct expression is obtained by first deriving $Y_l^{-l}(\theta,\varphi)$ and acting with the raising operator:

$$Y_l^m(\theta,\varphi) = \frac{(-1)^{l+m}}{2^l l!}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\exp(im\varphi)(\sin\theta)^{m}\frac{d^{l+m}}{d(\cos\theta)^{l+m}}(\sin\theta)^{2l}.$$

Also, the Spherical Harmonics with $m \geq 0$ may be written in the form

$$Y_l^m(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(-1)^m\exp(im\varphi)P_l^m(\cos\theta),$$

where $P_l^m(\xi)$ for $0 < m \leq l$ is an associated Legendre function of the first kind and, $P_l^0(\xi) \equiv P_l(\xi)$ is a Legendre polynomial. For $m < 0$, we can write

$$Y_l^m(\theta,\varphi) = (-1)^m[Y_l^{-m}(\theta,\varphi)]^*$$

to match up with our previous definitions.

Explicitly, the first few Spherical Harmonics are [Merzbacher Eq. 11.82]

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$$

$$Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta = \mp\sqrt{\frac{3}{8\pi}}\frac{x \pm iy}{r}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) = \sqrt{\frac{5}{16\pi}}\frac{2z^2 - x^2 - y^2}{r^2}$$

$$Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\cos\theta\sin\theta = \mp\sqrt{\frac{15}{8\pi}}\frac{(x \pm iy)z}{r^2}$$

$$Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}}e^{\pm 2i\varphi}\sin^2\theta = \sqrt{\frac{15}{32\pi}}\frac{(x \pm iy)^2}{r^2}$$

As eigenfunctions of Hermitian operators $(\mathbf{L}^2, \mathbf{L}_z)$ with distinct eigenvalues, the Spherical Harmonics form an orthonormal set,

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta\, [Y_l^m(\theta,\varphi)]^* Y_{l'}^{m'}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}.$$

It can be shown that they provide a complete basis for expansion of the angular component of $3D$ wave functions, with closure relation

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}[Y_l^m(\theta,\varphi)]^* Y_l^m(\theta',\varphi') = \delta(\hat{r},\hat{r}'),$$

where $\hat{r} \leftrightarrow (\theta,\varphi)$, $\hat{r}' \leftrightarrow (\theta',\varphi')$, and $\delta(\hat{r},\hat{r}')$ is the ‘solid-angle’ delta function defined by

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta F(\hat{r}')\delta(\hat{r},\hat{r}') = F(\hat{r})$$

for any angular function $F(\hat{r})$.