Ph195b lecture notes, 3/4/02

(The first part of this set should have been handed out earlier...)

Orbital angular momentum

As we saw last time, the angular momentum operator in QM is defined just the same way as in classical mechanics:

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$
$$= \vec{\mathbf{r}} \times \frac{\hbar}{i}\boldsymbol{\nabla},$$

where $\vec{\mathbf{r}} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $\vec{\mathbf{p}}$ are the 'vector' position and momentum operators. The cross-product notation has its usual meaning,

$$\mathbf{L}_x = \mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y,$$
$$\mathbf{L}_y = \mathbf{z}\mathbf{p}_x - \mathbf{x}\mathbf{p}_z,$$
$$\mathbf{L}_z = \mathbf{x}\mathbf{p}_y - \mathbf{y}\mathbf{p}_x.$$

Note that there are no subtleties here regarding operator ordering, since the product terms only involve operators acting on different 'kinetic' subspaces. Also, the angular momentum operators are clearly Hermitian.

Commutators of the angular momentum operators with position and momentum operators are easy to compute, *e.g.*,

$$[\mathbf{L}_x, \mathbf{x}] = [\mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y, \mathbf{x}] = 0,$$
$$[\mathbf{L}_x, \mathbf{y}] = [\mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y, \mathbf{y}] = -\mathbf{z}[\mathbf{p}_y, \mathbf{y}] = i\hbar\mathbf{z},$$
$$[\mathbf{L}_x, \mathbf{p}_y] = [\mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y, \mathbf{p}_y] = [\mathbf{y}, \mathbf{p}_y]\mathbf{p}_z = i\hbar\mathbf{p}_z,$$

etc. Hence, among the angular momentum operators themselves,

$$[\mathbf{L}_x, \mathbf{L}_y] = [\mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y, \mathbf{z}\mathbf{p}_x - \mathbf{x}\mathbf{p}_z]$$
$$= [\mathbf{y}\mathbf{p}_z, \mathbf{z}\mathbf{p}_x] + [\mathbf{z}\mathbf{p}_y, \mathbf{x}\mathbf{p}_z]$$
$$= \mathbf{y}\mathbf{p}_x[\mathbf{p}_z, \mathbf{z}] + \mathbf{p}_y\mathbf{x}[\mathbf{z}, \mathbf{p}_z]$$
$$= -i\hbar\mathbf{y}\mathbf{p}_x + i\hbar\mathbf{p}_y\mathbf{x}$$
$$= i\hbar\mathbf{L}_z,$$

and by cyclic permutation

$$[\mathbf{L}_x, \mathbf{L}_y] = i\hbar\mathbf{L}_z, \quad [\mathbf{L}_y, \mathbf{L}_z] = i\hbar\mathbf{L}_x, \quad [\mathbf{L}_z, \mathbf{L}_x] = i\hbar\mathbf{L}_y.$$

As we shall see over the next few lectures, this all-important set of commutation relations defines the structure of the angular momentum *algebra*. We already see that for quantum systems, no two Cartesian components of the angular momentum vector can be simultaneously precisely defined (unless $\langle\vec{\mathbf{L}}\rangle = 0$)! For example,

$$\Delta\mathbf{L}_x\Delta\mathbf{L}_y \geq \frac{\hbar}{2}|\langle\mathbf{L}_z\rangle|^2,$$

and so on.

Algebraic derivation of eigenvalues

Similar to the LHO, the commutation relations can be used as the basis of an algebraic approach to solving the angular momentum eigenvalue problem. Following Merzbacher, we switch notation in this section to $\vec{\mathbf{J}} \leftrightarrow \vec{\mathbf{L}}$,

$$[\mathbf{J}_x, \mathbf{J}_y] = i\hbar\mathbf{J}_z, \quad [\mathbf{J}_y, \mathbf{J}_z] = i\hbar\mathbf{J}_x, \quad [\mathbf{J}_z, \mathbf{J}_x] = i\hbar\mathbf{J}_y,$$

in order to emphasize that the derivation applies not only to orbital angular momentum but any set of operators (*e.g.* spin) that satisfy the given commutation relations.

First we define a few new operators in terms of the old ones,

$$\begin{aligned}
\mathbf{J}^2 &\equiv \vec{\mathbf{J}} \cdot \vec{\mathbf{J}} = \mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2,\\
\mathbf{J}_+ &\equiv \mathbf{J}_x + i\mathbf{J}_y,\\
\mathbf{J}_- &= \mathbf{J}_+^\dagger = \mathbf{J}_x - i\mathbf{J}_y.
\end{aligned}$$

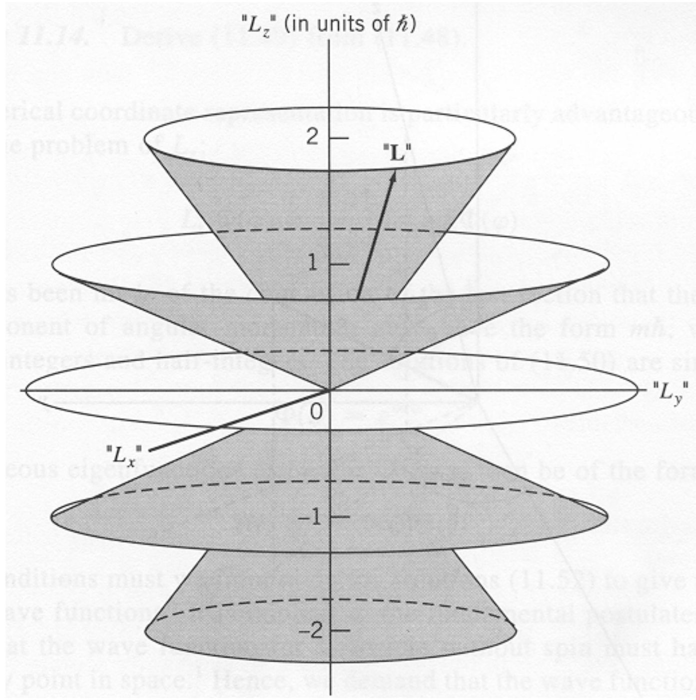
As you might guess from the notation, $\mathbf{J}_\pm$ will play similar roles to those of the annihilation and creation operators. These angular momentum raising and lowering operators are clearly not Hermitian, but the 'total angular momentum' operator $\mathbf{J}^2$ is Hermitian. Note that

$$\begin{aligned}
\left[\mathbf{J}^2, \mathbf{J}_x\right] &= \left[\mathbf{J}_y^2, \mathbf{J}_x\right] + \left[\mathbf{J}_z^2, \mathbf{J}_x\right]\\
&= \mathbf{J}_y^2\mathbf{J}_x - \mathbf{J}_x\mathbf{J}_y^2 + \mathbf{J}_z^2\mathbf{J}_x - \mathbf{J}_x\mathbf{J}_z^2\\
&= \mathbf{J}_y^2\mathbf{J}_x - (\mathbf{J}_y\mathbf{J}_x + i\hbar\mathbf{J}_z)\mathbf{J}_y + \mathbf{J}_z(\mathbf{J}_x\mathbf{J}_z + i\hbar\mathbf{J}_y) - \mathbf{J}_x\mathbf{J}_z^2\\
&= \mathbf{J}_y^2\mathbf{J}_x - \mathbf{J}_y\mathbf{J}_x\mathbf{J}_y - i\hbar\mathbf{J}_z\mathbf{J}_y + \mathbf{J}_z\mathbf{J}_x\mathbf{J}_z + i\hbar\mathbf{J}_z\mathbf{J}_y - \mathbf{J}_x\mathbf{J}_z^2\\
&= \mathbf{J}_y^2\mathbf{J}_x - \mathbf{J}_y(\mathbf{J}_y\mathbf{J}_x + i\hbar\mathbf{J}_z) + (\mathbf{J}_x\mathbf{J}_z + i\hbar\mathbf{J}_y)\mathbf{J}_z - \mathbf{J}_x\mathbf{J}_z^2\\
&= -i\hbar\mathbf{J}_y\mathbf{J}_z + i\hbar\mathbf{J}_y\mathbf{J}_z\\
&= 0,
\end{aligned}$$

and

$$\left[\mathbf{J}^2, \mathbf{J}_x\right] = \left[\mathbf{J}^2, \mathbf{J}_y\right] = \left[\mathbf{J}^2, \mathbf{J}_z\right] = 0$$

by symmetry. Hence, we find that while there are no states that have zero uncertainty for more than one component of $\vec{\mathbf{J}}$ (except those with $\langle\vec{\mathbf{J}}\rangle = 0$), there should exist simultaneous eigenstates of $\mathbf{J}^2$ and any one Cartesian component of angular momentum. If we think about this in classical terms, it's like saying that we can't precisely define more than one Cartesian component of the angular momentum vector, but we can precisely define its *length* together with any one Cartesian component. This line of reasoning leads to the 'semiclassical' picture of angular momentum [Merzbacher 11.2],



From this picture, we may also infer that for a given eigenvalue of $\mathbf{J}^2$, the corresponding eigenvalues of $\mathbf{J}_{x,y,z}$ must lie within a certain range.

Let's label the simultaneous eigenstates of $\mathbf{J}^2$ and $\mathbf{J}_z$ (for instance) by $|\lambda m\rangle$, with

$$\mathbf{J}^2|\lambda m\rangle = \lambda\hbar^2|\lambda m\rangle,$$
$$\mathbf{J}_z|\lambda m\rangle = m\hbar|\lambda m\rangle,$$

where we are making use of our previous insight that $\hbar$ is the natural unit of angular momentum. According to the semiclassical picture, we would like to think that $m^2 \leq \lambda$. To see that this is indeed true, we may write

$$\begin{aligned}\mathbf{J}^2 - \mathbf{J}_z^2 &= \mathbf{J}_x^2 + \mathbf{J}_y^2 = \frac{1}{4}\left[(\mathbf{J}_+ + \mathbf{J}_-)^2 - (\mathbf{J}_+ - \mathbf{J}_-)^2\right] \\ &= \frac{1}{2}(\mathbf{J}_+\mathbf{J}_- + \mathbf{J}_-\mathbf{J}_+) \\ &= \frac{1}{2}\left(\mathbf{J}_+\mathbf{J}_+^\dagger + \mathbf{J}_+^\dagger\mathbf{J}_+\right).\end{aligned}$$

Recall that the expectation value of an operator of the form $\mathbf{AA}^\dagger$ or $\mathbf{A}^\dagger\mathbf{A}$ must be non-negative since, *e.g.*,

$$\begin{aligned}\left\langle\Psi|\mathbf{A}^\dagger\mathbf{A}|\Psi\right\rangle &= (\mathbf{A}|\Psi\rangle)^\dagger(\mathbf{A}|\Psi\rangle) \\ &= |\mathbf{A}|\Psi\rangle|^2 \\ &\geq 0.\end{aligned}$$

Hence we conclude that

$$\begin{aligned}\left\langle\Psi|\mathbf{J}^2 - \mathbf{J}_z^2|\Psi\right\rangle &= \frac{1}{2}\left\langle\Psi|\mathbf{J}_+\mathbf{J}_+^\dagger|\Psi\right\rangle + \frac{1}{2}\left\langle\Psi|\mathbf{J}_+^\dagger\mathbf{J}_+|\Psi\right\rangle \\ &\geq 0\end{aligned}$$

for any state, and in particular for simultaneous eigenstates $|\lambda m\rangle$,

$$\langle \lambda m|\mathbf{J}^2 - \mathbf{J}_z^2|\lambda m\rangle = \lambda\hbar^2 - m^2\hbar^2 \geq 0,$$
$$\lambda \geq m^2.$$

For a given value of λ, we have thus established an allowed range for m. This may remind us of the procedure we used in the case of the LHO, where we found a lower bound on eigenvalues of the number operator and this gave us a starting point for deriving its eigenspectrum. Motivated by this analogy, we would next like to show that the angular momentum raising and lowering operators can be used to obtain new eigenstates from old eigenstates:

$$\begin{aligned}
\mathbf{J}_z|\lambda m\rangle &= m\hbar|\lambda m\rangle, \\
\mathbf{J}_z[\mathbf{J}_+|\lambda m\rangle] &= \mathbf{J}_z(\mathbf{J}_x + i\mathbf{J}_y)|\lambda m\rangle \\
&= (\mathbf{J}_x\mathbf{J}_z + i\hbar\mathbf{J}_y + i(\mathbf{J}_y\mathbf{J}_z - i\hbar\mathbf{J}_x))|\lambda m\rangle \\
&= [(\mathbf{J}_x + i\mathbf{J}_y)\mathbf{J}_z + \hbar(\mathbf{J}_x + i\mathbf{J}_y)]|\lambda m\rangle \\
&= (m+1)\hbar[\mathbf{J}_+|\lambda m\rangle], \\
\mathbf{J}_z[\mathbf{J}_-|\lambda m\rangle] &= \mathbf{J}_z(\mathbf{J}_x - i\mathbf{J}_y)|\lambda m\rangle \\
&= (m-1)\hbar[\mathbf{J}_-|\lambda m\rangle].
\end{aligned}$$

Also, $\mathbf{J}_\pm$ commutes with $\mathbf{J}^2$ since $\mathbf{J}_{x,y}$ do, so

$$\begin{aligned}
\mathbf{J}^2[\mathbf{J}_\pm|\lambda m\rangle] &= \mathbf{J}_\pm\left[\mathbf{J}^2|\lambda m\rangle\right] \\
&= \lambda\hbar^2[\mathbf{J}_\pm|\lambda m\rangle].
\end{aligned}$$

Hence either $\mathbf{J}_\pm|\lambda m\rangle = 0$ or $\mathbf{J}_\pm|\lambda m\rangle$ is *proportional to* a new simultaneous eigenstate $|\lambda(m\pm 1)\rangle$:

$$\begin{aligned}
\mathbf{J}_+|\lambda m\rangle &= C_+(\lambda, m)\hbar|\lambda(m+1)\rangle, \\
\mathbf{J}_-|\lambda m\rangle &= C_-(\lambda, m)\hbar|\lambda(m-1)\rangle,
\end{aligned}$$

where $C_\pm(\lambda, m)$ are normalization coefficients (which may be complex, in principle) yet to be determined.

Now we use the fact that there are lower and upper bounds on m (in terms of λ). Using j_λ to denote the maximum value of m for a given λ, we have

$$\mathbf{J}_+|\lambda j_\lambda\rangle = 0.$$

We would like to use this 'boundary condition' to derive j_λ in terms of λ, so let's try to put $\mathbf{J}^2$ and $\mathbf{J}_z$ on the LHS. An easy way to do this is to multiply from the left by $\mathbf{J}_-$, yielding

$$\begin{aligned}
0 &= \mathbf{J}_-\mathbf{J}_+|\lambda j_\lambda\rangle \\
&= (\mathbf{J}_x - i\mathbf{J}_y)(\mathbf{J}_x + i\mathbf{J}_y)|\lambda j_\lambda\rangle \\
&= \left(\mathbf{J}_x^2 + \mathbf{J}_y^2 + i\mathbf{J}_x\mathbf{J}_y - i\mathbf{J}_y\mathbf{J}_x\right)|\lambda j_\lambda\rangle \\
&= \left(\mathbf{J}^2 - \mathbf{J}_z^2 - \hbar\mathbf{J}_z\right)|\lambda j_\lambda\rangle \\
&= (\lambda - j_\lambda^2 - j_\lambda)\hbar^2|\lambda j_\lambda\rangle.
\end{aligned}$$

Hence we have

$$\lambda - j_\lambda^2 - j_\lambda = 0,$$
$$\lambda = j_\lambda(j_\lambda + 1).$$

Similarly, on the lower end we have

$$\mathbf{J}_-|\lambda j'_\lambda\rangle = 0,$$

where j'_λ denotes the minimum value of m for a given λ. This leads to

$$\begin{aligned}0 &= \mathbf{J}_+\mathbf{J}_-|\lambda j'_\lambda\rangle \\ &= \left(\mathbf{J}_x^2+\mathbf{J}_y^2 - i\mathbf{J}_x\mathbf{J}_y + i\mathbf{J}_y\mathbf{J}_x\right)|\lambda j_\lambda\rangle \\ &= \left(\mathbf{J}^2 - \mathbf{J}_z^2 + \hbar\mathbf{J}_z\right)|\lambda j_\lambda\rangle \\ &= (\lambda - j_\lambda^2 + j_\lambda)\hbar^2|\lambda j_\lambda\rangle,\end{aligned}$$

and

$$\lambda = j'_\lambda(j'_\lambda - 1).$$

The two conditions we have derived yield a consistency condition

$$j_\lambda(j_\lambda+1) = j'_\lambda(j'_\lambda - 1),$$

whose solutions are

$$j'_\lambda = -j_\lambda, \qquad j'_\lambda = j_\lambda + 1.$$

The latter solution is not allowed since we have defined $j_\lambda \geq j'_\lambda$, so we may henceforth assume that $j'_\lambda = -j_\lambda$.

It is important to note that the bounds j_λ, j'_λ we have derived are *exact*, since for example $\mathbf{J}_-|\lambda m\rangle = 0$ is satisfied *only* by $m = j'_\lambda$. Suppose we now start with the lowest eigenstate $|\lambda j'_\lambda\rangle$ and repeatedly apply the raising operator $\mathbf{J}_+$. We saw above that this leads (up to normalization) to new eigenstates

$$(\mathbf{J}_+)^n|\lambda j'_\lambda\rangle \propto |\lambda(j'_\lambda+n)\rangle$$

until the upper bound on m is exceeded. Again, this ladder will truncate properly only if ends precisely at $|\lambda j_\lambda\rangle$, since $\mathbf{J}_+|\lambda m\rangle = 0$ is satisfied only by $m = j_\lambda$. Hence we conclude that $j_\lambda = j'_\lambda + n = -j_\lambda + n$ for some integer $n \geq 0$, or equivalently

$$2j_\lambda = n.$$

This shows that j_λ *can only be a non-negative integer or half-integer*. For a given λ, and thus given j_λ, we see that the possible eigenvalues of J_z are

$$m\hbar = j_\lambda\hbar,\ (j_\lambda-1)\hbar,\ (j_\lambda-2),\ \cdots - (j_\lambda-1)\hbar,\ -j_\lambda\hbar$$

are likewise all integral multiples of $\hbar$ (for integer j_λ) or half-integral multiples of $\hbar$ (for half-integer j_λ). In general, there are $2j_\lambda+1$ allowed values of m for a given λ. Hence, we may think of the eigenstates $|j;m\rangle$ as being arranged in a 'tiered' structure:

$$\begin{array}{lllll}|0;0\rangle \\ |1/2;-1/2\rangle & |1/2;1/2\rangle \\ |1;-1\rangle & |1;0\rangle & |1;1\rangle \\ |3/2;-3/2\rangle & |3/2;-1/2\rangle & |3/2;1/2\rangle & |3/2;3/2\rangle \\ |2;-2\rangle & |2;-1\rangle & |2;0\rangle & |2;1\rangle & |2;2\rangle \\ \vdots\end{array}$$

Up to this point we have thought of λ and m as the natural quantum numbers to specify angular momentum eigenstates, since we started by considering simultaneous eigenstates of $\mathbf{J}^2$ and $\mathbf{J}_z$. Noting that λ and j_λ are in one-to-one correspondence, however, we can just as well label eigenstates according to m and j, with $\lambda = j(j+1)$. This is in fact the more common convention, where j is often referred to as 'the angular momentum' of a state $|jm\rangle$ and m is its 'azimuthal quantum number.'

From this point of view, quantum mechanics gives us the following picture:

1. Angular momentum is quantized. The angular momentum state space breaks down into subspaces of fixed j, where j must be an integer or half-integer (including zero). The subspace with angular momentum j has $2j+1$ basis states.
2. No two Cartesian components of the angular momentum vector $\langle \vec{J} \rangle$ can be simultaneously specified without uncertainty.
3. Any one Cartesian component $m\hbar$ may be precisely specified together with j.
4. The 'length' of the angular momentum vector $\sqrt{\langle \mathbf{J}^2 \rangle} = \hbar\sqrt{j(j+1)}$ corresponds to $\hbar j$ only in the limit $j \to \infty$.
5. The length $\hbar\sqrt{j(j+1)}$ is therefore always greater than any single Cartesian component $\hbar m$, with the difference made up by 'fluctuations' in the orthogonal components.

Finally, let us compute the normalization coefficients $C_\pm(\lambda, m)$, defined above. This will allow us to contemplate matrix representations of the angular momentum operators. For normalization we require

$$\begin{aligned} ||\lambda(m\pm 1)\rangle|^2 &= \left[\hbar^2 |C_\pm(\lambda,m)|^2\right]^{-1} (\mathbf{J}_\pm|\lambda m\rangle)^\dagger (\mathbf{J}_\pm|\lambda m\rangle) = 1, \\ \hbar^2|C_\pm(\lambda,m)|^2 &= \langle \lambda m|\mathbf{J}_\mp \mathbf{J}_\pm|\lambda m\rangle \\ &= \langle \lambda m|(\mathbf{J}_x \mp i\mathbf{J}_y)(\mathbf{J}_x \pm i\mathbf{J}_y)|\lambda m\rangle \\ &= \left\langle \lambda m|\mathbf{J}^2 - \mathbf{J}_z^2 \mp \hbar \mathbf{J}_z|\lambda m\right\rangle \\ &= (\lambda - m^2 \mp m)\hbar^2. \end{aligned}$$

Hence we may set

$$\begin{aligned} |C_\pm(\lambda,m)|^2 &= \lambda - m^2 \mp m \\ &= j(j+1) - m^2 \mp m \\ &= j(j+1) - m(m\pm 1). \end{aligned}$$

It is customary to set the phase of these coefficients to zero. We may then write (with a little rearranging)

$$\begin{aligned} C_+(\lambda,m) &= \sqrt{(j-m)(j+m+1)}, \\ C_-(\lambda,m) &= \sqrt{(j+m)(j-m+1)}. \end{aligned}$$

We thus see that, *within* a given j subspace, the raising and lowering operators may be represented as $(2j+1)$-dimensional matrices with real nonzero elements only on the first super- or sub-diagonal. Since all the angular momentum operators can in fact be given as linear combinations of the raising and lowering operators, we may conclude that the j subspaces are in fact closed with respect to operation of the angular momentum operators. As we'll discuss in coming lectures, this important insight will help us understand addition of angular momenta in terms of linear representations of the angular momentum algebra.

(This part picks up where we left off...)

Representation theory recap

An algebraic group is a collection of elements with a multiplication table, such that associativity and closure hold, there exists an identity element, and every element in the group has an inverse. We like to define groups so that we can derive useful theorems about them.

The set of rotations of Euclidean 3-space form a group, if multiplication is taken to correspond to the successive application of rotations. For example, if we let $R_{\hat{n}}(\varphi)$ denote a clockwise rotation by angle φ about the unit vector $\hat{n}$,

$$R_{y}(\pi)R_{\hat{x}}(\pi) = R_{\hat{z}}(\pi).$$

A linear representation is a mapping between elements of a group and (square) matrices

$$a \mapsto D(a),$$

such that

$$D(a)D(b) = D(ab).$$

Two representations related by a linear change of basis

$$D(a) \mapsto S^{-1}D(a)S$$

are said to be equivalent. As an example, consider the dihedral group D_2

$$\begin{matrix} e & a & b & c \\ a & e & c & b \\ b & c & e & a \\ c & b & a & e \end{matrix}$$

and the following representation on $\mathbf{R}^2$

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$D(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Under the change of basis

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which is described by the matrix

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

we find

$$D(e) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{aligned}
D(a) &\mapsto \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
D(b) &\mapsto \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
D(c) &\mapsto \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}$$

Hence we see that the representations

$$e \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b \leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c \leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$e \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b \leftrightarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad c \leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

are equivalent.

Bigger representations can be constructed from smaller ones by direct sum or direct product – we'll get back to this later!

For *any* set of three Hermitian operators $\{\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z\}$ satisfying the 'angular momentum' commutation relations

$$[\mathbf{J}_x, \mathbf{J}_y] = i\hbar\mathbf{J}_z, \quad [\mathbf{J}_y, \mathbf{J}_z] = i\hbar\mathbf{J}_x, \quad [\mathbf{J}_z, \mathbf{J}_x] = i\hbar\mathbf{J}_y,$$

the association

$$(\hat{n}, \varphi) \mapsto \exp\left(\frac{-i}{\hbar}\hat{n} \cdot \vec{\mathbf{J}}\varphi\right),$$

defines a linear representation of the rotation group $R(3)$.

Based on algebraic properties of angular momentum operators, we determined (several lectures ago) that $\mathbf{J}_z$ and the operator

$$\mathbf{J}^2 \equiv \mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2$$

have simultaneous eigenstates denoted $|j;m\rangle$, where

$$\mathbf{J}^2|j;m\rangle = j(j+1)\hbar^2|j;m\rangle,$$

$$\mathbf{J}_z|j;m\rangle = m\hbar|j;m\rangle.$$

Here m takes on values in the range $-j, -j+1, -j+2, \ldots\ j-1, j$ and j is restricted to non-negative integer or half-integer values. Because of the relations

$$\mathbf{J}_x = \frac{1}{2}(\mathbf{J}_+ + \mathbf{J}_-), \qquad \mathbf{J}_y = \frac{-i}{2}(\mathbf{J}_+ - \mathbf{J}_-),$$

where

$$\mathbf{J}_\pm|j;m\rangle = \sqrt{(j \mp m)(j \pm m+1)}\,\hbar|j;m \pm 1\rangle,$$

we see that subspaces of constant j are invariant under action of the angular momentum operators $\{\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z\}$. In fact, this means that any linear combination

$$c_x\mathbf{J}_x + c_y\mathbf{J}_y + c_z\mathbf{J}_z$$

leaves the subspaces invariant (where $c_{x,y,z}$ are arbitrary complex coefficients), as indeed do arbitrary “multinomials” such as

$$5\mathbf{J}_x^4 + 2i\mathbf{J}_y\mathbf{J}_z^3 + \mathbf{J}_z\mathbf{J}_x.$$

In particular, subspaces of constant j are invariant under action of the rotation operators

$$\exp\left(\frac{-i}{\hbar}\hat{n} \cdot \vec{\mathbf{J}}\varphi\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-i\varphi}{\hbar}\right)^k(n_x\mathbf{J}_x + n_y\mathbf{J}_y + n_z\mathbf{J}_z)^k.$$

We may therefore conclude that the restrictions of angular momentum operators to subspaces of constant j generate subrepresentations of $R(3)$. In particular, if we choose the conventional ordering of basis states

$$|j;m\rangle = |0;0\rangle, \left|\frac{1}{2};+\frac{1}{2}\right\rangle, \left|\frac{1}{2};-\frac{1}{2}\right\rangle, |1;+1\rangle, |1;0\rangle, |1;-1\rangle, \left|\frac{3}{2};+\frac{3}{2}\right\rangle, \left|\frac{3}{2};+\frac{1}{2}\right\rangle, \ldots$$

then the matrices corresponding to $\{\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z\}$ in this basis are

$$\mathbf{J}_x \leftrightarrow \hbar\begin{pmatrix} 0 & & & & & & \\ & 0 & \frac{1}{2} & & & & \\ & \frac{1}{2} & 0 & & & & \cdots \\ & & & 0 & 2^{-1/2} & 0 & \\ & & & 2^{-1/2} & 0 & 2^{-1/2} & \\ & & & 0 & 2^{-1/2} & 0 & \\ & & \vdots & & & & \ddots \end{pmatrix},$$

$$\mathbf{J}_y \leftrightarrow \hbar\begin{pmatrix} 0 & & & & & & \\ & 0 & -\frac{i}{2} & & & & \\ & \frac{i}{2} & 0 & & & & \cdots \\ & & & 0 & -i2^{-1/2} & 0 & \\ & & & i2^{-1/2} & 0 & -i2^{-1/2} & \\ & & & 0 & i2^{-1/2} & 0 & \\ & & \vdots & & & & \ddots \end{pmatrix},$$

$$\mathbf{J}_z \leftrightarrow \hbar\begin{pmatrix} 0 & & & & & & \\ & +\frac{1}{2} & 0 & & & & \\ & 0 & -\frac{1}{2} & & & & \cdots \\ & & & +1 & 0 & 0 & \\ & & & 0 & 0 & 0 & \\ & & & 0 & 0 & -1 & \\ & & \vdots & & & & \ddots \end{pmatrix}.$$

Here the blank entries are all equal to zero. We can immediately pick out generators of the subrepresentations with dimension 1×1 $(j = 0)$, 2×2 $(j = 1/2)$, 3×3 $(j = 1)$, etc, and we see that the infinite-dimensional representation of $R(3)$ decomposes as the direct sum of finite-dimensional irreps. We can thus generate a perfectly good 3×3 representation of the rotation group by

$$(\hat{n}, \varphi) \mapsto \exp\left(\frac{-i}{\hbar}\hat{n} \cdot \vec{\mathbf{J}}\varphi\right)$$

with, *e.g.*,

$$\mathbf{J}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \mathbf{J}_z = \hbar\begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is straightforward to verify that these are Hermitian and satisfy the angular momentum commutation relations.

Addition of angular momenta

Let's say that we have quantum system A with $j = 1/2$, whose state lives in the two-dimensional Hilbert space H_A. The angular momentum operators on this space will be

$$\mathbf{J}_x^A = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{J}_y^A = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{J}_z^A = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let's say we also have quantum system B that lives in the three-dimensional Hilbert space H_B with $j = 1$. The angular momentum operators on this space will be

$$\mathbf{J}_x^B = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{J}_y^B = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \mathbf{J}_z^B = \hbar\begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

If these systems are coupled, the joint state lives in the tensor product Hilbert space

$$H_{AB} = H_A \otimes H_B.$$

In the natural product basis for the joint space, we have matrix representations

$$\mathbf{J}_x^A \otimes \mathbf{1}^B = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{1}^A \otimes \mathbf{J}_x^B = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{J}_y^A \otimes \mathbf{1}^B = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{1}^A \otimes \mathbf{J}_y^B = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \end{pmatrix},$$

$$\mathbf{J}_z^A \otimes \mathbf{1}^B = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{1}^A \otimes \mathbf{J}_z^B = \hbar\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

As we mentioned last time, taking the tensor product of angular-momentum Hilbert spaces generates a direct-product representation of $R(3)$:

$$\begin{aligned}(\hat{n},\varphi) \mapsto\ & \exp\left(\frac{-i}{\hbar}\hat{n}\cdot\vec{\mathbf{J}}^A\varphi\right)\otimes\exp\left(\frac{-i}{\hbar}\hat{n}\cdot\vec{\mathbf{J}}^B\varphi\right)\\ =\ & \exp\left(\frac{-i}{\hbar}\hat{n}\cdot\left(\vec{\mathbf{J}}^A\otimes\mathbf{1}^B+\mathbf{1}^A\otimes\vec{\mathbf{J}}^B\right)\varphi\right)\\ \equiv\ & \exp\left(\frac{-i}{\hbar}\hat{n}\cdot\vec{\mathbf{J}}\varphi\right),\end{aligned}$$

where we have defined

$$\vec{\mathbf{J}}\equiv\vec{\mathbf{J}}^A\otimes\mathbf{1}^B+\mathbf{1}^A\otimes\vec{\mathbf{J}}^B,$$

meaning

$$\begin{aligned}\mathbf{J}_x &= \mathbf{J}_x^A\otimes\mathbf{1}^B+\mathbf{1}^A\otimes\mathbf{J}_x^B\\ &= \frac{\hbar}{2}\begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{pmatrix}+\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0&0&0&0\\1&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&1\\0&0&0&0&1&0\end{pmatrix}\\ &= \hbar\begin{pmatrix}0&2^{-1/2}&0&\frac{1}{2}&0&0\\2^{-1/2}&0&2^{-1/2}&0&\frac{1}{2}&0\\0&2^{-1/2}&0&0&0&\frac{1}{2}\\\frac{1}{2}&0&0&0&2^{-1/2}&0\\0&\frac{1}{2}&0&2^{-1/2}&0&2^{-1/2}\\0&0&\frac{1}{2}&0&2^{-1/2}&0\end{pmatrix},\end{aligned}$$

$$\mathbf{J}_y = \mathbf{J}_y^A \otimes \mathbf{1}^B + \mathbf{1}^A \otimes \mathbf{J}_y^B$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \end{pmatrix} + \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \end{pmatrix}$$

$$= \hbar\begin{pmatrix} 0 & -i2^{-1/2} & 0 & -\frac{i}{2} & 0 & 0 \\ i2^{-1/2} & 0 & -i2^{-1/2} & 0 & -\frac{i}{2} & 0 \\ 0 & i2^{-1/2} & 0 & 0 & 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 & 0 & 0 & -i2^{-1/2} & 0 \\ 0 & \frac{i}{2} & 0 & i2^{-1/2} & 0 & -i2^{-1/2} \\ 0 & 0 & \frac{i}{2} & 0 & i2^{-1/2} & 0 \end{pmatrix},$$

$$\mathbf{J}_z = \mathbf{J}_z^A \otimes \mathbf{1}^B + \mathbf{1}^A \otimes \mathbf{J}_z^B$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} + \hbar\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \hbar\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}.$$

Clearly, J_x and J_y are not block-diagonal. Yet, we may expect from dimension arithmetic that the $2 \times 3 = 6$-dimensional joint space $H_A \otimes H_B$ might decompose into the direct sum of 2 $(j = 1/2)$ and 4 $(j = 3/2)$ dimensional spaces. How can we check this?

Recall that $\mathbf{J}^2$ is a Casimir operator for representations of the rotation group:

$$\mathbf{J}^2 \equiv \mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2,$$

$$\left[\mathbf{J}^2, \vec{\mathbf{J}}\right] = 0.$$

Verification of the second line (that $\mathbf{J}^2$ commutes with the generators) is left as an exercise. More importantly, we move on to computing $\mathbf{J}^2$ in matrix form (still working in the product basis),

$$\mathbf{J}_x^2 = \hbar^2 \begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{5}{4} & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ \frac{1}{2} & 0 & \frac{3}{4} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{3}{4} & 0 & \frac{1}{2} \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{5}{4} & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2} & 0 & \frac{3}{4} \end{pmatrix},$$

$$\mathbf{J}_y^2 = \hbar^2 \begin{pmatrix} \frac{3}{4} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{5}{4} & 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2} & 0 & \frac{3}{4} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{3}{4} & 0 & -\frac{1}{2} \\ -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{5}{4} & 0 \\ 0 & -\frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2} & 0 & \frac{3}{4} \end{pmatrix},$$

$$\mathbf{J}_z^2 = \hbar^2 \begin{pmatrix} \frac{9}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{4} \end{pmatrix},$$

$$\mathbf{J}^2 = \begin{pmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{11}{4} & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \frac{7}{4} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \frac{7}{4} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & \frac{11}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{4} \end{pmatrix}.$$

This is clearly not proportional to the identity, so Schur's Second Lemma tells us that $H_A \otimes H_B$ can be decomposed. Recall that to do this, we should diagonalize $\mathbf{J}^2$ to put it into the form

$$\begin{pmatrix} \lambda_1 I_1 & & \\ & \lambda_2 I_2 & \\ & & \ddots \end{pmatrix}.$$

If we ask Maple to solve for eigenvalues and eigenvectors of $\mathbf{J}^2$, we get

$$\left\{ \begin{matrix} 0 \\ 1 \\ 0 \\ -\sqrt{2} \\ 0 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \\ -\frac{1}{2}\sqrt{2} \\ 0 \end{matrix} \right\} \leftrightarrow \frac{3}{4}, \left\{ \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{matrix}, \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 0 \\ \frac{1}{2}\sqrt{2} \\ 0 \\ 1 \\ 0 \end{matrix}, \begin{matrix} 0 \\ \sqrt{2} \\ 0 \\ 1 \\ 0 \\ 0 \end{matrix} \right\} \leftrightarrow \frac{15}{4}.$$

Hence, there is indeed a subspace of dimension 2 and eigenvalue $3/4$ (corresponding to $j = 1/2$), and another of dimension 4 and eigenvalue $15/4$ (corresponding to $j = 3/2$). The transformation matrix into the eigenbasis of $\mathbf{J}^2$ is simply obtained by 'stacking together' the normalized eigenvectors of $\mathbf{J}^2$. Doing so in a judicious order, we obtain

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 3^{-1/2} & 0 & 0 & \sqrt{2/3} & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 0 & \sqrt{1/3} & 0 \\ -\sqrt{2/3} & 0 & 0 & 3^{-1/2} & 0 & 0 \\ 0 & -\sqrt{1/3} & 0 & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S^{-1} = S^T = \begin{pmatrix} 0 & 3^{-1/2} & 0 & -\sqrt{2/3} & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 & -\sqrt{1/3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 3^{-1/2} & 0 & 0 \\ 0 & 0 & \sqrt{1/3} & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and finally

$$
\begin{aligned}
\mathbf{J}_x &\mapsto S^{-1}\mathbf{J}_x S \\
&= \hbar \begin{pmatrix} 0 & 3^{-1/2} & 0 & -\sqrt{2/3} & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 & -\sqrt{1/3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 3^{-1/2} & 0 & 0 \\ 0 & 0 & \sqrt{1/3} & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 2^{-1/2} & 0 & \frac{1}{2} & 0 & 0 \\ 2^{-1/2} & 0 & 2^{-1/2} & 0 & \frac{1}{2} & 0 \\ 0 & 2^{-1/2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 2^{-1/2} & 0 \\ 0 & \frac{1}{2} & 0 & 2^{-1/2} & 0 & 2^{-1/2} \\ 0 & 0 & \frac{1}{2} & 0 & 2^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 3^{-1/2} & 0 & 0 & \sqrt{2/3} & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 0 & \sqrt{1/3} & 0 \\ -\sqrt{2/3} & 0 & 0 & 3^{-1/2} & 0 & 0 \\ 0 & -\sqrt{1/3} & 0 & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \hbar \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2}\sqrt{3} \\ 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{3} & 0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{J}_y \mapsto\ & S^{-1}\mathbf{J}_y S \\
=\ & \hbar\begin{pmatrix} 0 & 3^{-1/2} & 0 & -\sqrt{2/3} & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 & -\sqrt{1/3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 3^{-1/2} & 0 & 0 \\ 0 & 0 & \sqrt{1/3} & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
& \times\begin{pmatrix} 0 & -i2^{-1/2} & 0 & -\frac{i}{2} & 0 & 0 \\ i2^{-1/2} & 0 & -i2^{-1/2} & 0 & -\frac{i}{2} & 0 \\ 0 & i2^{-1/2} & 0 & 0 & 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 & 0 & 0 & -i2^{-1/2} & 0 \\ 0 & \frac{i}{2} & 0 & i2^{-1/2} & 0 & -i2^{-1/2} \\ 0 & 0 & \frac{i}{2} & 0 & i2^{-1/2} & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 3^{-1/2} & 0 & 0 & \sqrt{2/3} & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 0 & \sqrt{1/3} & 0 \\ -\sqrt{2/3} & 0 & 0 & 3^{-1/2} & 0 & 0 \\ 0 & -\sqrt{1/3} & 0 & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
=\ & \hbar\begin{pmatrix} 0 & -\frac{1}{2}i & 0 & 0 & 0 & 0 \\ \frac{1}{2}i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}i\sqrt{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2}i\sqrt{3} & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & -\frac{1}{2}i\sqrt{3} \\ 0 & 0 & 0 & 0 & \frac{1}{2}i\sqrt{3} & 0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{J}_z &\mapsto S^{-1}\mathbf{J}_z S \\
&= \hbar\begin{pmatrix} 0 & 3^{-1/2} & 0 & -\sqrt{2/3} & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 & -\sqrt{1/3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 3^{-1/2} & 0 & 0 \\ 0 & 0 & \sqrt{1/3} & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&\quad\times\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 3^{-1/2} & 0 & 0 & \sqrt{2/3} & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 0 & \sqrt{1/3} & 0 \\ -\sqrt{2/3} & 0 & 0 & 3^{-1/2} & 0 & 0 \\ 0 & -\sqrt{1/3} & 0 & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \hbar\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}.
\end{aligned}
$$

Magic! Hopefully it is now clear that our tensor-product representation on $H_A \otimes H_B$ is equivalent to the direct sum of a $j = 1/2$ and a $j = 3/2$ irrep.

Let's try to understand why this is useful. Say we have the following Hamiltonian that couples our two systems:

$$
\mathbf{H} = \gamma \vec{\mathbf{J}}^A \cdot \vec{\mathbf{J}}^B,
$$

where γ is a real constant. Since

$$
\begin{aligned}
\left[\mathbf{H}, \mathbf{J}^2\right] &= \gamma\left[\vec{\mathbf{J}}^A \cdot \vec{\mathbf{J}}^B, \left(\mathbf{J}^A\right)^2 \otimes \mathbf{1}^B + \mathbf{1}^A \otimes \left(\mathbf{J}^B\right)^2 + 2\vec{\mathbf{J}}^A \cdot \vec{\mathbf{J}}^B\right] \\
&= \gamma\left[\mathbf{J}_x^A \otimes \mathbf{J}_x^B + \mathbf{J}_y^A \otimes \mathbf{J}_y^B + \mathbf{J}_z^A \otimes \mathbf{J}_z^B, \left(\mathbf{J}^A\right)^2 \otimes \mathbf{1}^B\right] \\
&\quad + \gamma\left[\mathbf{J}_x^A \otimes \mathbf{J}_x^B + \mathbf{J}_y^A \otimes \mathbf{J}_y^B + \mathbf{J}_z^A \otimes \mathbf{J}_z^B, \mathbf{1}^A \otimes \left(\mathbf{J}^B\right)^2\right] \\
&= 0,
\end{aligned}
$$

since $\left[\vec{\mathbf{J}}^A, \left(\mathbf{J}^A\right)^2\right] = \left[\vec{\mathbf{J}}^B, \left(\mathbf{J}^B\right)^2\right] = 0$. We see that total j is a good quantum number, *i.e.*, it is possible to choose the energy eigenstates such that they are also eigenstates of $\mathbf{J}^2$. This in turn means that we need only solve the problem of diagonalizing **H** *within* the subspaces of constant j, rather than on the entire state space $H_A \otimes H_B$ at once – that is, we only need to perform a 2×2 and a 4×4 diagonalization, rather than a 6×6. Plunging ahead, we can first calculate **H** in the product basis,

$$
\begin{aligned}
\mathbf{H} &= \gamma \vec{\mathbf{J}}^A \cdot \vec{\mathbf{J}}^B \\
&= \gamma \left(\mathbf{J}_x^A \otimes \mathbf{J}_x^B + \mathbf{J}_y^A \otimes \mathbf{J}_y^B + \mathbf{J}_z^A \otimes \mathbf{J}_z^B \right) \\
&= \gamma\hbar^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 2^{-3/2} & 0 \\ 0 & 0 & 0 & 2^{-3/2} & 0 & 2^{-3/2} \\ 0 & 0 & 0 & 0 & 2^{-3/2} & 0 \\ 0 & 2^{-3/2} & 0 & 0 & 0 & 0 \\ 2^{-3/2} & 0 & 2^{-3/2} & 0 & 0 & 0 \\ 0 & 2^{-3/2} & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\quad + \gamma\hbar^2 \begin{pmatrix} 0 & 0 & 0 & 0 & -2^{-3/2} & 0 \\ 0 & 0 & 0 & 2^{-3/2} & 0 & -2^{-3/2} \\ 0 & 0 & 0 & 0 & 2^{-3/2} & 0 \\ 0 & 2^{-3/2} & 0 & 0 & 0 & 0 \\ -2^{-3/2} & 0 & 2^{-3/2} & 0 & 0 & 0 \\ 0 & -2^{-3/2} & 0 & 0 & 0 & 0 \end{pmatrix} + \gamma\hbar^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \\
&= \gamma\hbar^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},
\end{aligned}
$$

which when transformed to the direct sum basis yields

$$\gamma\hbar^2\begin{pmatrix} 0 & 3^{-1/2} & 0 & -\sqrt{2/3} & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 & -\sqrt{1/3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 3^{-1/2} & 0 & 0 \\ 0 & 0 & \sqrt{1/3} & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 3^{-1/2} & 0 & 0 & \sqrt{2/3} & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 0 & \sqrt{1/3} & 0 \\ -\sqrt{2/3} & 0 & 0 & 3^{-1/2} & 0 & 0 \\ 0 & -\sqrt{1/3} & 0 & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \gamma\hbar^2\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

So indeed, the energy spectrum has only two distinct components: a $j = 1/2$ 'doublet' with energy $-\gamma\hbar^2$ and a $j = 3/2$ 'quartet' with energy $\gamma\hbar^2/2$.

Note that it might have turned out that **H** would only be block-diagonal when transformed to the direct sum basis, in which case we would have to diagonalize each of the blocks to get the energy eigenvalues.

Clebsch-Gordon coefficients

The transformation matrix S^{-1} that we defined above gives the mapping from a product basis

$$|j_A;m_A\rangle\otimes|j_B;m_B\rangle = \left\{\begin{matrix} |\tfrac{1}{2};+\tfrac{1}{2}\rangle\otimes|1;+1\rangle, |\tfrac{1}{2};+\tfrac{1}{2}\rangle\otimes|1;0\rangle, |\tfrac{1}{2};+\tfrac{1}{2}\rangle\otimes|1;-1\rangle, \\ |\tfrac{1}{2};-\tfrac{1}{2}\rangle\otimes|1;+1\rangle, |\tfrac{1}{2};-\tfrac{1}{2}\rangle\otimes|1;0\rangle, |\tfrac{1}{2};-\tfrac{1}{2}\rangle\otimes|1;-1\rangle \end{matrix}\right\}$$

to the direct sum basis

$$|j;m\rangle = \left\{\begin{matrix} |\tfrac{1}{2};+\tfrac{1}{2}\rangle, |\tfrac{1}{2};-\tfrac{1}{2}\rangle, \\ |\tfrac{3}{2};+\tfrac{3}{2}\rangle, |\tfrac{3}{2};+\tfrac{1}{2}\rangle, |\tfrac{3}{2};-\tfrac{1}{2}\rangle, |\tfrac{3}{2};-\tfrac{3}{2}\rangle, \end{matrix}\right\}.$$

In particular, we note that the product basis state

$$\left| \frac{1}{2};-\frac{1}{2} \right\rangle \otimes |1;0\rangle \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

maps to the vector

$$S^{-1}\left| \frac{1}{2};-\frac{1}{2} \right\rangle \otimes |1;0\rangle = \begin{pmatrix} 0 & 3^{-1/2} & 0 & -\sqrt{2/3} & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 & -\sqrt{1/3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 3^{-1/2} & 0 & 0 \\ 0 & 0 & \sqrt{1/3} & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -\sqrt{1/3} \\ 0 \\ 0 \\ \sqrt{2/3} \\ 0 \end{pmatrix}$$

$$= -\frac{1}{\sqrt{3}}\left| \frac{1}{2};\,-\frac{1}{2} \right\rangle + \sqrt{\frac{2}{3}}\left| \frac{3}{2};\,-\frac{1}{2} \right\rangle.$$

The coefficients that relate product basis vectors to direct-sum basis vectors

$$\begin{aligned} (\langle j_A;m_A| \otimes \langle j_B;m_B|)|j;m\rangle &\equiv \langle j_A j_B;m_A m_B|j;m\rangle \\ &\equiv C(j_A j_B;m_A m_B|j;m) \end{aligned}$$

are known as *Clebsch-Gordon* or *Wigner* or *vector-addition* or *vector-coupling coefficients*. Everyone has a different notation for these things, so it is important to understand ***what*** they are so that you will be able to decipher various notations from context. In our example, we have just shown that

$$C\left(\frac{1}{2}1;\,-\frac{1}{2}0\middle|\frac{1}{2};\,-\frac{1}{2}\right) = -\frac{1}{\sqrt{3}},$$
$$C\left(\frac{1}{2}1;\,-\frac{1}{2}0\middle|\frac{3}{2};\,-\frac{1}{2}\right) = \sqrt{\frac{2}{3}},$$
$$C\left(\frac{1}{2}1;\,-\frac{1}{2}0\middle|\frac{1}{2};\,+\frac{1}{2}\right) = 0,$$
$$C\left(\frac{1}{2}1;\,-\frac{1}{2}0\middle|\frac{3}{2};\,+\frac{3}{2}\right) = 0,$$
$$C\left(\frac{1}{2}1;\,-\frac{1}{2}0\middle|\frac{3}{2};\,+\frac{1}{2}\right) = 0,$$
$$C\left(\frac{1}{2}1;\,-\frac{1}{2}0\middle|\frac{3}{2};\,-\frac{3}{2}\right) = 0,$$

Clebsch-Gordon (C-G) coefficients vanish unless the ‘triangular inequality’

$$|j_A - j_B| \le j \le j_A + j_B$$

and the requirement

$$m_A + m_B = m$$

are *both* fulfilled. We’ll look at an interpretation of these conditions next time.

The C-G coefficients also have important symmetries, such as

$$\begin{aligned} C(ab;\alpha\beta|c,\alpha+\beta) &= (-1)^{a+b-c}C(ba;\beta\alpha|c,\alpha+\beta) \\ &= (-1)^{a+b-c}C(ab;\,-\alpha,-\beta|c,\,-\alpha-\beta) \\ &= C(ba;\,-\beta,\,-\alpha|c,\,-\alpha-\beta) \end{aligned}$$

where we have inserted some commas for clarity.