Ph195a

Study problems to prepare for class on Wednesday October 3.

You are not required to hand in a write-up of these problems. It is not essential that you work these problems through to a final correct answer, but you should come to class prepared to present a coherent approach to solving each one. As a rough guide, you are meant to spend about six hours preparing (reading notes and working problems) for each class meeting.

Questions 0-3 refer to the lecture notes for 1 October (handed out at the first class meeting):

0. Make sure you understand what is meant by the sentence that follows equation 8.
1. Try to understand the relation between the formal definition of a projection operator (equation 12) and your intuitive notion of what projection means in a vector space.

2. Try to use basic rules of calculus to derive equation 14 from equation 13, using the Taylor expansion in equation 15 as an intermediate step.

3. Unitary evolution with a time-independent Hamiltonian corresponds to rotation of the state space around a fixed axis. If the Hamiltonian has a non-trivial nullspace (of dimension 1, 2, 3, ...), what can be said about the "geometric" relation between this axis of rotation and the nullspace?

Questions 4-6 are based on lecture notes for both 1 & 3 October

4. Use the decomposition into diagonal form ($M = S\Lambda S^{-1}$) to compute

$$\exp\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

5. Use the operator spectral decomposition ($\mathbf{O} = \sum_i \lambda_i \mathbf{P}_i$) to prove that all the eigenvalues of a unitary operator are unimodular, *i.e.*,

$$|\lambda_i|^2 = 1.$$

6. Assume we have a quantum system that lives in a two-dimensional Hilbert space, and that we have chosen a pair of orthonormal basis kets $|x\rangle$ and $|y\rangle$. Let's say we're playing a game in which our friend Charlie secretly prepares the state of this system to be in one of the following two states:

$$|\Psi_a\rangle = |x\rangle,$$
$$|\Psi_b\rangle = \cos\beta|x\rangle + \sin\beta|y\rangle.$$

Here β is a real-valued angle between 0 and 90 degrees. Our task is to perform a measurement and use the result to guess which state Charlie prepared.

Find the standard measurement $\{\mathbf{P}_A, \mathbf{P}_B\}$, with $\mathbf{P}_A\mathbf{P}_B = 0$ and $\mathbf{P}_A + \mathbf{P}_B = \mathbf{1}$, that minimizes the average probability of error ζ :

$$\zeta = \frac{1}{2}[\langle\Psi_b|P_A|\Psi_b\rangle + \langle\Psi_a|P_B|\Psi_a\rangle].$$

Hint: A standard measurement in two dimensions is completely specified by a single ket $|A\rangle$. Then $\mathbf{P}_A$ is the projector onto $|A\rangle$ and $\mathbf{P}_B$ is the projector onto a ket $|B\rangle$ that is orthogonal to $|A\rangle$. Since the states Ψ_a and Ψ_b are guarenteed to have real coefficients, so will $|A\rangle$ and $|B\rangle$. Hence, you can assume

$$|A\rangle = \cos\alpha|x\rangle + \sin\alpha|y\rangle$$

and simply optimize over the real-valued angle α.