## Study problems for 10/10

0. In equation (4) from the lecture notes for 10/3, explain the difference between the strict meaning of the angled-brackets on the left and right hand sides of the equation

$$\langle q^n \rangle = \langle \mathbf{O}^n \rangle.$$

1. Suppose we have a two-dimensional quantum system, prepared in the initial state

$$|\Psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is not known whether the Hamiltonian for this system corresponds to $\mathbf{H}_A = \hbar\omega_A \boldsymbol{\sigma}_x$ or $\mathbf{H}_B = \hbar\omega_B \boldsymbol{\sigma}_x$, where

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\omega_{A,B}$ are real-valued scalars.
We would like to determine which Hamiltonian is correct, by letting the system evolve for a time $\tau$ and then performing a measurement on the system state $|\Psi(\tau)\rangle$. What is the earliest time $\tau_0$ at which this determination can be made with zero probability of error?

2. For this problem we'll work in a three-dimensional Hilbert space, with basis kets $|x\rangle$, $|y\rangle$, and $|z\rangle$. Suppose we prepare the following ensemble of quantum states:

$$\begin{aligned} 50\% \quad & |x\rangle, \\ 20\% \quad & \frac{1}{\sqrt{2}}[|x\rangle + |y\rangle], \\ 20\% \quad & \frac{1}{\sqrt{2}}[|x\rangle + |z\rangle], \\ 10\% \quad & \frac{1}{\sqrt{2}}[|z\rangle - |y\rangle]. \end{aligned}$$

Find the ensemble of three mutually-orthogonal states whose density operator is identical to that of the above four-membered ensemble. (Note that it's okay to use computer software for this problem)

4. In a two-dimensional Hilbert space with basis kets $|x\rangle$ and $|y\rangle$, let

$$\begin{aligned} |\Psi_1\rangle &= \frac{1}{\sqrt{2}}[|x\rangle + |y\rangle], \\ \rho_1 &= |\Psi_1\rangle\langle\Psi_1|, \\ \rho_2 &= \frac{1}{2}[|x\rangle\langle x| + |y\rangle\langle y|]. \end{aligned}$$

Find a standard measurement by which $\rho_1$ can sometimes be distinguished from $\rho_2$ with absolute certainty.

5. Merzbacher, exercise 10.6: Show that if **A** and **B** are commuting observables, then **B** has no nonvanishing matrix elements between eigenstates corresponding to different eigenvalues of **A**. In other words, if

$$\mathbf{A}|i^A\rangle = \lambda_i^A|i^A\rangle,$$
$$\mathbf{A}|j^A\rangle = \lambda_j^A|j^A\rangle,$$

and $\lambda_i^A \neq \lambda_j^A$, then

$$\langle j^A|\mathbf{B}|i^A\rangle = 0.$$

If **B** is a Hamiltonian operator, what can we infer about the time evolution of moments of **A**?

6. In finite dimensions, is it possible to have a pair of normal operators whose commutator is proportional to the identity?