Ph195 – Study problems for 10/17/01

1. Show that

$$\left[\mathbf{1}^A \otimes \mathbf{B}, \mathbf{A} \otimes \mathbf{1}^B\right] = 0,$$

where **B** is an arbitrary operator on subsystem B and **A** is an arbitrary operator on subsystem A.

2. Let H_A and H_B be a pair of two-dimensional Hilbert spaces, with given orthonormal bases $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$. Let $|\Psi_{AB}\rangle$ be the state

$$|\Psi_{AB}\rangle = \cos\theta\,|0_A\rangle \otimes |0_B\rangle + \sin\theta\,|1_A\rangle \otimes |1_B\rangle.$$

For $0 < \theta < \frac{\pi}{2}$, this is an entangled state.

The *purity* ζ of $\tilde{\boldsymbol{\rho}}_A = \mathrm{Tr}_B[|\Psi_{AB}\rangle\langle\Psi_{AB}|]$,

$$\zeta \equiv \mathrm{Tr}[\tilde{\boldsymbol{\rho}}_A^2],$$

is a good measure of the 'entangledness' of states in H_{AB}. For pure states of the above form, find the extrema of ζ with respect to θ ($0 \le \zeta \le \frac{\pi}{2}$). Do entangled states have large ζ or small ζ?

3. Again let H_A and H_B be a pair of two-dimensional Hilbert spaces, with given orthonormal bases $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$. Consider the 'controlled-not' operator on H_{AB},

$$\mathbf{U}_{AB} = \mathbf{P}_0^A \otimes \mathbf{1}^B + \mathbf{P}_1^A \otimes \boldsymbol{\sigma}_x^B.$$

Here $\mathbf{P}_0^A = |0_A\rangle\langle 0_A|$, $\mathbf{P}_1^A = |1_A\rangle\langle 1_A|$, and $\boldsymbol{\sigma}_x^B = |0_B\rangle\langle 1_B| + |1_B\rangle\langle 0_B|$.

First, write a matrix representation for $\mathbf{U}_{AB}$ with respect to the following (ordered) basis for H_{AB}:

$$|0_A\rangle \otimes |0_B\rangle, \quad |0_A\rangle \otimes |1_B\rangle, \quad |1_A\rangle \otimes |0_B\rangle, \quad |1_A\rangle \otimes |1_B\rangle.$$

Find the eigenvectors of $\mathbf{U}_{AB}$ – you should be able to do this by inspection. Do any of them correspond to entangled states?

4. Working with the same system as in problems 3NS and 4NS, find a factorizable 'input' state

$$|\Psi_{AB}^{in}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$$

such that the 'output' state

$$|\Psi_{AB}^{out}\rangle = \mathbf{U}_{AB}|\Psi_{AB}^{in}\rangle$$

is *maximally entangled*. That is, find any factorizable $|\Psi_{AB}^{in}\rangle$ such that $\mathrm{Tr}[\tilde{\boldsymbol{\rho}}_A^2] = \frac{1}{2}$, where

$$\tilde{\boldsymbol{\rho}}_A = \mathrm{Tr}_B[|\Psi_{AB}^{out}\rangle\langle\Psi_{AB}^{out}|].$$