

**Ph195a Lecture Notes 12/5/01**  
**(Andrew Doherty)**

**Steady-state solutions of the Bloch Equations at zero temperature**

Recall the Bloch Equations with dissipation,

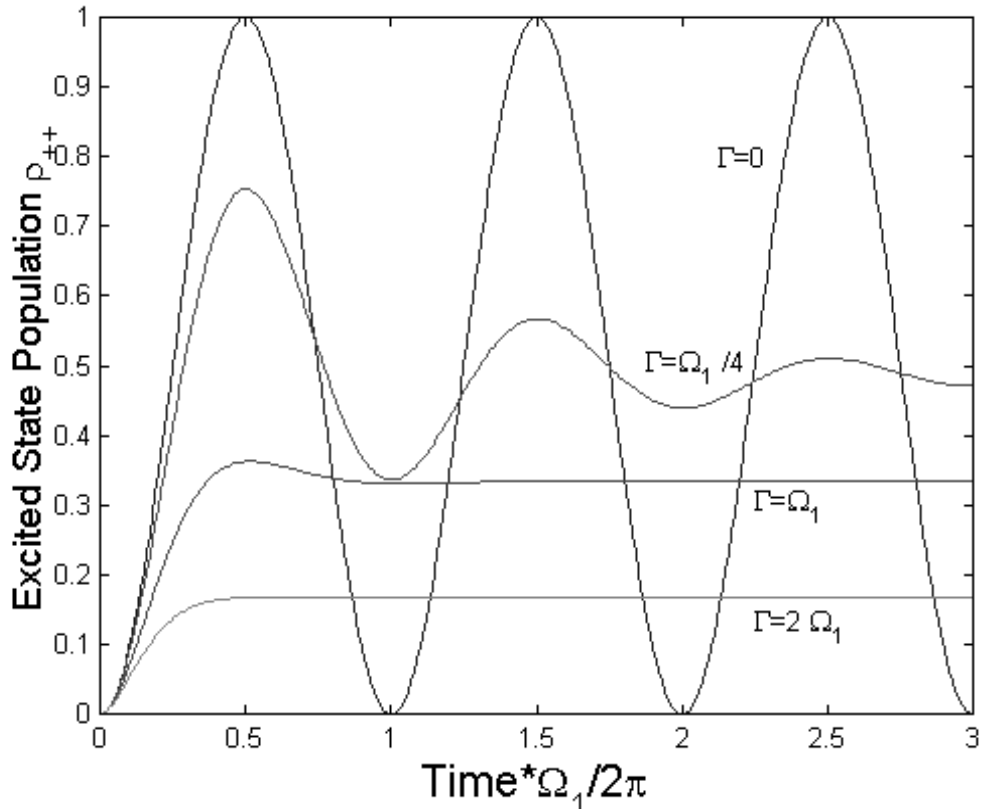
$$\begin{aligned}\dot{v}_x &= -\Delta v_y - v_x T_2^{-1}, \\ \dot{v}_y &= \Delta v_x + \gamma b_1 v_z - v_y T_2^{-1}, \\ \dot{v}_z &= -\gamma b_1 v_y - (v_z - v_z^0) T_1^{-1},\end{aligned}$$

Today we are going to look at some properties of its steady-state solutions in the “zero-temperature” limit  $T_1^{-1} \equiv \Gamma = 2T_2^{-1}$ . Note that  $\Gamma$  thus represents the decay rate of the excited state. Qualitatively, this corresponds to a limit in which  $v_z^0 \sim -1$  and the steady-state Bloch vector in the absence of driving ( $b_1 = 0$ ) is simply  $\vec{v} = (0, 0, -1)$ . In third term we’ll see how to obtain the Bloch Equations in this limit starting from a model of two-level quantum systems coupled to an actual zero-temperature reservoir.

In these terms, and defining  $\Omega_1 \equiv -\gamma b_1$ , we have

$$\begin{aligned}\dot{v}_x &= -\Delta v_y - \frac{\Gamma}{2} v_x, \\ \dot{v}_y &= \Delta v_x - \Omega_1 v_z - \frac{\Gamma}{2} v_y, \\ \dot{v}_z &= \Omega_1 v_y - \Gamma v_z - \Gamma.\end{aligned}$$

The following figures will give you an idea of the effect of damping on the dynamics of the two level system.



The oscillations between the ground and excited state are damped and for large enough  $\Gamma$  they disappear altogether. The excited state population settles down to a steady state value which depends  $\Omega_1, \Gamma, \Delta$  so long as  $\Gamma \neq 0$ . We see from the graph that the excited state population at long times is higher for larger  $\Omega_1$  or lower  $\Gamma$ .

To find the steady-state solutions we set all the left-hand sides (time derivatives) to zero:

$$v_x^{ss} = \frac{-2\Delta}{\Gamma} v_y^{ss},$$

$$v_y^{ss} = -\left(\frac{2\Delta^2}{\Gamma} + \frac{\Gamma}{2}\right)^{-1} \Omega_1 v_z^{ss},$$

$$v_z^{ss} = -\left(\frac{2\Delta^2}{\Gamma} + \frac{\Gamma}{2}\right)^{-1} \frac{\Omega_1^2}{\Gamma} v_z^{ss} - 1,$$

from which we obtain

$$\begin{aligned}
v_z^{ss} &= \frac{-1}{1 + \left(\frac{2\Delta^2}{\Gamma} + \frac{\Gamma}{2}\right)^{-1} \frac{\Omega_1^2}{\Gamma}} \\
&= \frac{-1}{1 + \frac{2\Gamma}{4\Delta^2 + \Gamma^2} \frac{\Omega_1^2}{\Gamma}} \\
&= \frac{-4\Delta^2 - \Gamma^2}{4\Delta^2 + \Gamma^2 + 2\Omega_1^2}, \\
v_y^{ss} &= \frac{\Omega_1}{\frac{2\Delta^2}{\Gamma} + \frac{\Gamma}{2} + \frac{\Omega_1^2}{\Gamma}} \\
&= \frac{2\Omega_1\Gamma}{4\Delta^2 + \Gamma^2 + 2\Omega_1^2}, \\
v_x^{ss} &= \frac{-4\Omega_1\Delta}{4\Delta^2 + \Gamma^2 + 2\Omega_1^2}.
\end{aligned}$$

From here we can easily compute the following two quantities of interest,

$$\begin{aligned}
\rho_{++}^{ss} &= \frac{1}{2}(v_z^{ss} + 1) \\
&= \frac{\Omega_1^2}{4} \frac{1}{\Delta^2 + (\Gamma^2/4) + (\Omega_1^2/2)}, \\
\varphi^{ss} &= \tan^{-1}\left(\frac{v_x^{ss}}{v_y^{ss}}\right) \\
&= \tan^{-1}\left(\frac{-2\Delta}{\Gamma}\right).
\end{aligned}$$

Let us look first at the former quantity  $\rho_{++}^{ss}$ , which is the steady-state population of the “excited” state  $|+_z\rangle$ . In terms of the **saturation parameter**  $s$  defined by

$$s \equiv \frac{\Omega_1^2/2}{\Delta^2 + (\Gamma^2/4)},$$

we have

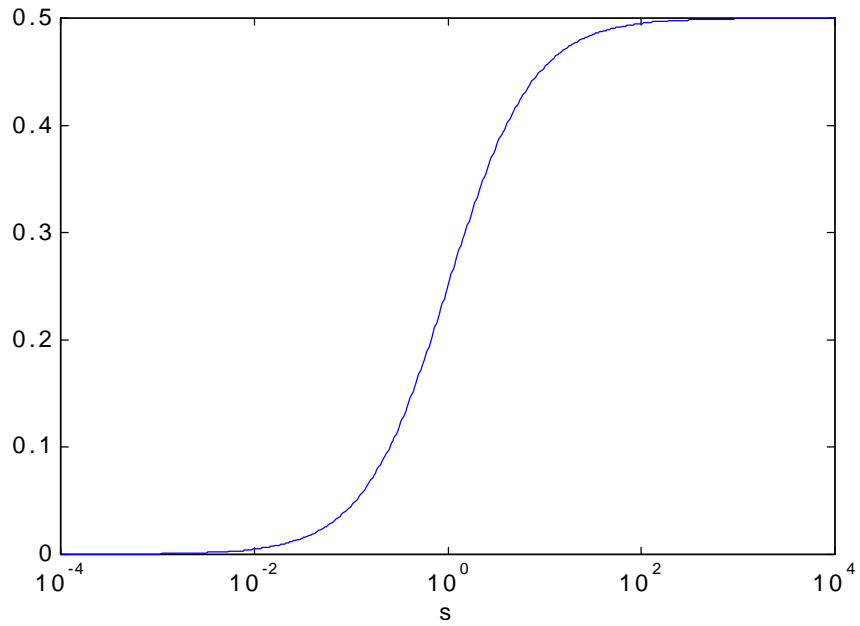
$$\rho_{++}^{ss} = \frac{1}{2} \frac{s}{1+s}.$$

Clearly  $s$  is a measure of the strength of the driving field  $\Omega_1$  relative to a weighted average of the detuning  $\Delta$  and decay rate  $\Gamma$ . In the weak driving limit ( $s \ll 1$ ) we have

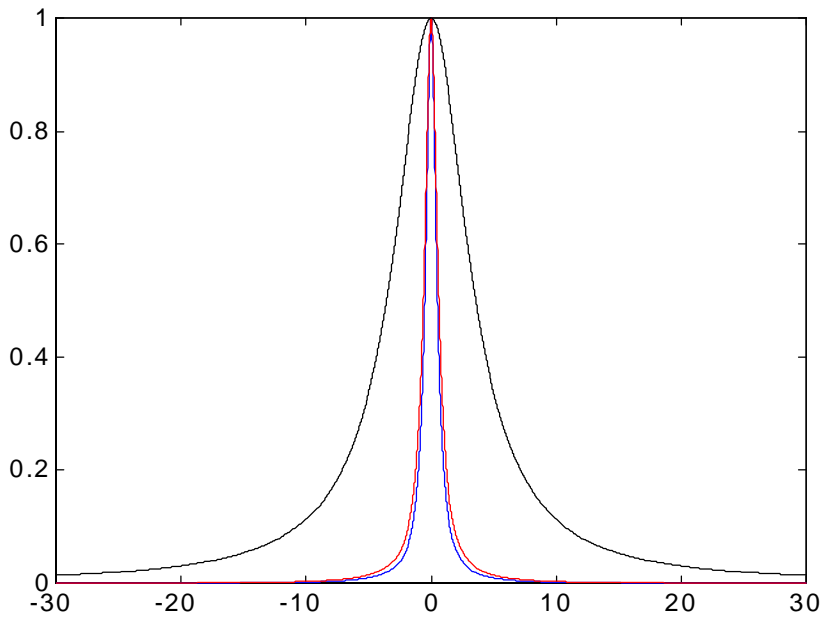
$$\rho_{++}^{ss} \approx \frac{1}{2}s = \frac{\Omega_1^2/4}{\Delta^2 + (\Gamma^2/4)},$$

meaning that a plot of the steady-state excitation vs. detuning is a Lorentzian with maximum amplitude  $(\Omega_1/\Gamma)^2$ , at  $\Delta = 0$ , and with FWHM  $\Gamma$ . Inspection of the exact expression above for  $\rho_{++}^{ss}$  reveals that if  $\Omega_1^2/2$  is not small compared to  $\Delta^2 + (\Gamma^2/4)$ , then the FWHM is increased and we have the effect known as **power broadening**.

Here’s a plot of the “saturation curve,” depicting  $\rho_{++}^{ss}$  as a function of  $s$ .



Now here's a plot of  $\rho_{++}^{ss}$  as a function of detuning, for several values of  $2\Omega_1^2/\Gamma^2$  (which is  $s$  at  $\Delta = 0$ ):



Here the narrowest curve is for  $s = 10^{-4}$ , then  $s = 1$ , then  $s = 100$ . Note that the presence of dissipation (even for very low power) means that the resonance is not infinitely narrow – we say that the resonance is broadened. Since each atom is coupled in the same way to a thermal bath we say that this resonance is *homogeneously broadened*.

## Free Induction Decay

(Cowan, Nuclear Magnetic Resonance and Relaxation, Ch 1, 3, 4. Allen and Eberly, Optical Resonance and Two-Level Atoms, Ch 9)

In this section we will imagine a simple nuclear magnetic resonance (NMR) experiment although the phenomena we will discuss are generic to many physical systems, including, for example optical spectroscopy. As we go along some of the main features of NMR experiments will become clear and I will eventually discuss some of the main applications. Later in the lecture we will discuss molecular beam experiments which rely on rather similar physics.

In the NMR experiments a sample of some molecule or atom, whether solid, liquid or gas, is subjected to a magnetic field  $B_0$ . The atoms are assumed to have a gyromagnetic ratio  $\gamma$  due to their nuclear magnetic moment and because we know about two-state systems we will assume for simplicity that the nucleus is spin-1/2 and so each atom is a simple two state system. In reality the magnetic field felt by each atom will be slightly different because it is impossible to make the magnetic field  $B_0$  completely uniform over the whole sample.

In particular each different value of the magnetic field strength  $B_0$  corresponds to a different value of the Larmor frequency  $\omega_L = -\gamma B_0$  for the precession of the nuclear magnetic dipole. The different oscillation frequencies of each of the atoms means that the precession of the nuclear dipoles will dephase in a very similar way to the dephasing caused by dissipation. To be specific consider a large number of atoms in a magnetic field directed along the  $z$ -axis which has an average field strength  $\bar{B}_0$ , the corresponding Larmor frequency is of course  $\bar{\omega}_L = -\gamma\bar{B}_0$ . Each atom though sees a, possibly different, field  $B_0$  and will thus precess around the  $z$ -axis at the frequency  $\omega_L = -\gamma B_0$ . This means that the resonance of atomic sample will be broadened since for different frequencies of a rotating magnetic field  $b_1$  different atoms will be in resonance. Since this broadening is because different atoms are resonant at different frequencies and not because of some dissipation process that affects each atom equally we term this *inhomogenous broadening*.

We can define the detuning  $\Delta = \omega_L - \bar{\omega}_L$  and consider the dynamics in a frame rotating at the average Larmor frequency  $\bar{\omega}_L$ . Recall that the Hamiltonian in this rotating frame is

$$H = \frac{1}{2}\hbar(\Delta\sigma_z - \gamma b_1(t)\sigma_x)$$

where  $b_1$  is the strength of a small rotating magnetic field perpendicular to the  $z$ -axis

$$\vec{B}_\perp(t) = b_1(t)(\bar{x}\cos(\bar{\omega}_L t) + \bar{y}\sin(\bar{\omega}_L t)).$$

Note that the perturbing magnetic field is applied at the average Larmor frequency.

This leads to the Bloch equations (ignoring dissipation for the moment)

$$\begin{aligned}\dot{v}_{x\Delta} &= -\Delta v_{y\Delta}, \\ \dot{v}_{y\Delta} &= \Delta v_{x\Delta} + \gamma b_1 v_{z\Delta}, \\ \dot{v}_{z\Delta} &= -\gamma b_1 v_{y\Delta}.\end{aligned}$$

In the rotating frame the field  $b_1$  tends to rotate the Bloch vector about the  $x$ -axis while the

detuning of each atom leads to a precession around the  $z$ -axis. I am using  $\Delta$  subscripts to indicate that the Bloch vector describes the state of one of the atoms in the ensemble experiencing a particular magnetic field strength  $B_0$  not the Bloch vector corresponding to the total density matrix.

Imagine that initially all the dipoles point along the magnetic fields and thus

$$\rho(0) = |-z\rangle\langle -z|$$

or  $v_x = 0 = v_y, v_z = -1$ . Then imagine turning on a strong field  $b_1$  such that  $\gamma b_1 \gg \Delta$  and we can ignore the precession of the atoms during the magnetic field pulse. If we leave the field on for a time  $t_1 = \pi/(2\gamma b_1)$  then the Bloch vector completes a  $\pi/2$  rotation about the  $x$ -axis preparing all of the atoms in a state

$$\rho(t_1) = |-y\rangle\langle -y|$$

or  $v_x = 0 = v_z, v_y = -1$ . (This is often termed a  $\pi/2$ -pulse for obvious reasons.)

If the rotating field  $b_1$  is now left off the atoms will precess (remember that this is all in the rotating frame) about the  $z$ -axis. It is clear from the Bloch equations with  $b_1 = 0$  that each atom will be in the state

$$\begin{aligned} v_{x\Delta}(t) &= \sin[\Delta(t - t_1)], \\ v_{y\Delta}(t) &= -\cos[\Delta(t - t_1)], \\ v_{z\Delta}(t) &= 0. \end{aligned}$$

Thus  $v_{x\Delta} + iv_{y\Delta} = -i \exp[i\Delta(t - t_1)]$ .

Now each atom precesses at a different rate depending on the local magnetic field, we are interested in the state of the whole ensemble of atoms since this is what could be detected in an experiment. This is to say that we want to know what the density matrix expectation values are. Recall that the density matrix is

$$\rho = \sum_j p_j |\Psi_j\rangle\langle\Psi_j|$$

where in this case  $|\Psi_j\rangle$  is the state of an atom in the  $j$ th value of the magnetic field. Recall also that the Bloch vector corresponding to  $\rho$  is just the vector sum of the Bloch vectors of the individual atoms weighted by the probabilities  $p_j$ . So for example

$$v_x = \sum_j p_j v_{xj}$$

where  $v_{xj}$  is the  $x$ -component of the Bloch vector of an atom experiencing the  $j$ th value of the magnetic field.

Since each atom sees a different value of the magnetic field  $B_0$  each atom has a different detuning  $\Delta$ . Lets assume that the distribution of magnetic fields is a normal (Gaussian) distribution, this means that the distribution of detunings is also Gaussian and we can define a time that we will call  $T_2^*$  for reasons that will shortly become obvious and then say that the distribution of detunings is

$$p(\Delta) = \frac{T_2^*}{\pi} \exp(-\Delta^2 T_2^{*2}/\pi).$$

This is a Gaussian distribution of mean zero and variance  $\pi/2T_2^{*2}$ . I don't have any particularly good reason for assuming that the distribution is Gaussian except that this turns out to be a good approximation in many cases (for example samples at low temperature) and because the maths turns out to be simple in this case. In any case basically the same thing will happen regardless of the exact distribution of magnetic field strengths. Rather than worrying about individual values of  $p_j$  I am assuming a continuous distribution described by  $p(\Delta)$  and I replace the sum above by an integral. Then the average dipole is given by

$$\begin{aligned} v_x + iv_y &= \int_{-\infty}^{\infty} p(\Delta)(v_{x\Delta} + iv_{y\Delta})d\Delta \\ &= -i\frac{T_2^*}{\pi} \int_{-\infty}^{\infty} \exp(-\Delta^2 T_2^{*2}/\pi) \exp(i\Delta(t - t_1))d\Delta. \end{aligned}$$

This integral is one that you have probably seen before and leads to

$$v_x + iv_y = -i \exp\left[-\frac{\pi}{4} \left(\frac{t - t_1}{T_2^*}\right)^2\right].$$

Notice that the average dipole decays to zero in a time of the order of  $T_2^*$  as a result of the dephasing of the individual atomic dipoles caused by the different magnetic fields experienced by different atoms. We now see that the inhomogenous broadening of the transition in the frequency domain leads to the decay of the atomic magnetization in the time domain. The decay of the atomic magnetization is termed *free induction decay*.

Why free induction decay? Recall that one way to detect the precession of atomic dipole is to measure the induced EMF caused by the changing magnetisation of the atoms. We'll see in a minute that this induction signal dies away as the sample's magnetization does. The word free refers to the fact that the induction signal is measured without any rotating magnetic field  $b_1$  applied to the sample.

To describe the behaviour of the induced current we need to move back out of the rotating frame. Remember that we are working in rotating co-ordinates

$$\begin{aligned} \bar{x}'(t) &= \bar{x} \cos \bar{\omega}t + \bar{y} \sin \bar{\omega}_L t = \text{Re}[(\bar{x} + i\bar{y})e^{-i\bar{\omega}_L t}] \\ \bar{y}'(t) &= -\bar{x} \sin \bar{\omega}t + \bar{y} \cos \bar{\omega}_L t = \text{Im}[(\bar{x} + i\bar{y})e^{-i\bar{\omega}_L t}]. \end{aligned}$$

Thus

$$\bar{x}'(t) + i\bar{y}'(t) = (\bar{x} + i\bar{y})e^{-i\bar{\omega}_L t}$$

and

$$\bar{x} + i\bar{y} = (\bar{x}'(t) + i\bar{y}'(t))e^{i\bar{\omega}_L t}.$$

And so the magnetization of the atomic sample is

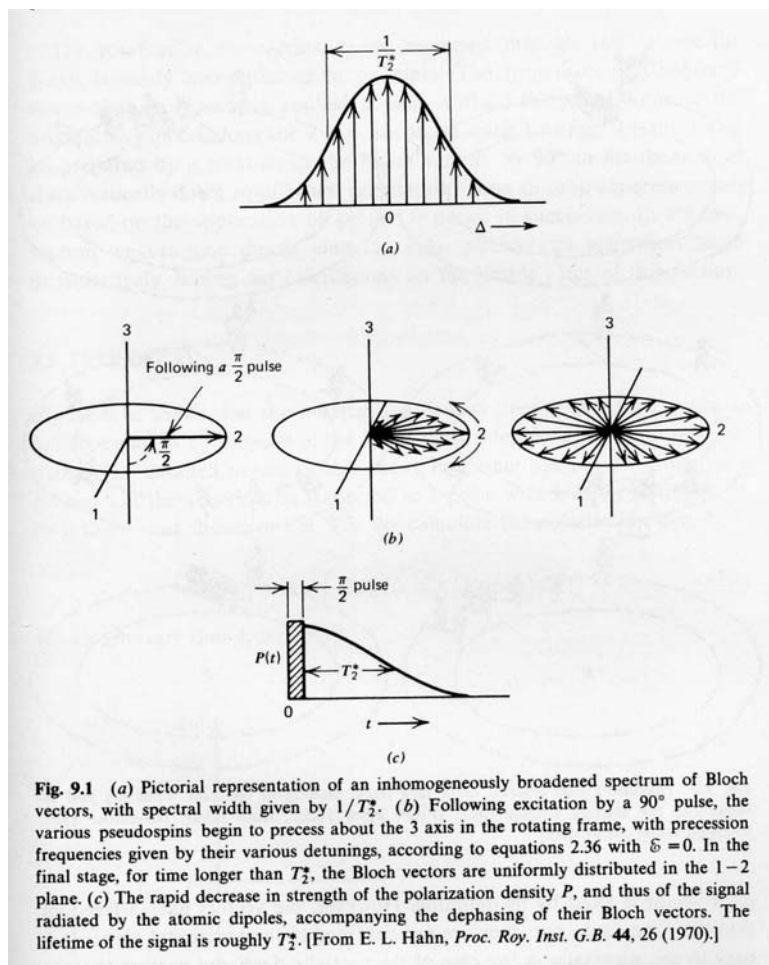
$$\begin{aligned} \vec{M} &= -\gamma \langle \vec{\mathbf{S}} \rangle \\ &= -\frac{\hbar\gamma}{2} \begin{pmatrix} \text{Re}[(v_x + iv_y)e^{i\bar{\omega}_L t}] \\ \text{Im}[(v_x + iv_y)e^{i\bar{\omega}_L t}] \\ v_z \end{pmatrix}. \end{aligned}$$

It is clear that in our example the magnetization of the atoms will initially oscillate in the  $x - y$  plane and decay away over a time of the order of  $T_2^*$ . If pick-up coils surrounding the atomic sample are oriented in the  $x - z$  plane then the time variation of the  $y$ -component of the magnetisation will lead to a time variation of the magnetic flux through the coils and thus to an induced current. This induced current will be proportional to the rate of change of the  $y$ -component of the magnetization

$$\begin{aligned}
 I &\propto \frac{d}{dt} \text{Im}[(v_x + iv_y)e^{i\bar{\omega}_L t}] \\
 &= -\frac{d}{dt} \exp\left[-\frac{\pi}{4} \left(\frac{t-t_1}{T_2^*}\right)^2\right] \cos(\bar{\omega}_L t) \\
 &= \exp\left[-\frac{\pi}{4} \left(\frac{t-t_1}{T_2^*}\right)^2\right] \left[ \bar{\omega}_L \sin(\bar{\omega}_L t) + \frac{\pi}{2T_2^{*2}}(t-t_1) \cos(\bar{\omega}_L t) \right]
 \end{aligned}$$

so the induced current too will decay away as the atomic dipoles dephase in the spatially varying magnetic field.

The free induction decay is pictorially represented in this page from the book by Allen and Eberly.



Note that we have ignored here the effect of the intrinsic decay described by  $T_2$ . It would be

a simple application of the Bloch equations that we started the lecture with to show that  $T_2$  will also contribute to the decay of the free induction current.

The time  $T_2^*$  is traditionally defined as a measure of the timescale of the decay of the free induction current, it has contributions from the variation of the magnetic field  $B_0$  as discussed above and from the intrinsic dephasing of the sample described by  $T_2$ . In liquid samples for example it is often the case that the variation of the magnetic field is by far the largest contribution to  $T_2^*$ . As we will see in the next section if magnetic field inhomogeneities are the main cause of the free induction decay then the disappearance of the sample's magnetisation is in fact reversible.

We have now described the mechanism of a simple pulsed NMR experiment. Note that the induced current tells us not only the value of  $T_2^*$  but also the Larmor frequency  $\bar{\omega}_L$ . In early NMR experiments this kind of measurement (and also CW NMR experiments which measure induced currents in time-varying but not pulsed magnetic fields) led to very precise measurements of gyromagnetic ratios  $\gamma$  which could often be related to the fundamental gyromagnetic ratios of protons or neutrons through theories about nuclear structure which in this way were experimentally tested. Equally though the Larmor frequency gives information about the local magnetic field  $B_0$  of the nuclei, presuming the gyromagnetic ratio is known. In more complex molecules this has very wide application in chemistry where nuclei in different parts of a molecule experience different local magnetic fields, over and above any experimentally applied field, which may be measured in this way. This provides evidence of a molecule's structure. Finally the most modern applications of NMR techniques are to medical imaging (often called MRI — magnetic resonance imaging). In this case it is the density of a particular spin (say a hydrogen nucleus) that is mapped out. In order to achieve this it is necessary to use spatially varying magnetic fields and the resolution of the technique depends on the field gradients that are available. This has an advantage over X-Ray techniques in that it is less harmful and more able to image soft tissue (for example your brain!)

## Spin Echo

It is a remarkable fact that if the dephasing that causes free induction decay is due to inhomogeneities in the applied magnetic field it may be reversed and the magnetisation of the atomic sample recovered.

To see how this might be the case let's think about the dynamics in the rotating frame again. In this frame each atom is precessing completely deterministically around the  $z$ -axis at some frequency  $\Delta$ . Since the frequencies are all different the atoms soon spread out evenly on the  $x - y$  plane of the Bloch sphere and the Bloch vector soon averages to zero. However if we could reverse the direction of precession of *all* of the atoms individually at some time  $t_2$  then they would start going backwards and would all line up again at time  $t_2 + (t_2 - t_1)$ .

We *can* reverse the direction of precession of all of the atoms by performing a rotation of  $180^\circ$  about the  $x$ -axis (a  $\pi$ -pulse).

To see this in detail lets go back to our calculation of the evolution of the state of an individual atom in the rotating frame.

At  $t_2$  we have that

$$\begin{aligned}v_{x\Delta}(t_2) &= \sin(\Delta(t_2 - t_1)), \\v_{y\Delta}(t_2) &= -\cos(\Delta(t_2 - t_1)), \\v_{z\Delta}(t_2) &= 0.\end{aligned}$$

Once again we will switch on a rotating field described by  $b_1$  which we will again assume to be very strong. We will leave this field on until time  $t_3$  where  $t_3 - t_2 = \pi/(\gamma b_1)$  and assuming that  $b_1$  is sufficiently strong that we can ignore  $\Delta$  in this time we have

$$\begin{aligned}v_{x\Delta}(t_3) &= \sin(\Delta(t_3 - t_1)), \\v_{y\Delta}(t_3) &= \cos(\Delta(t_3 - t_1)), \\v_{z\Delta}(t_3) &= 0.\end{aligned}$$

Again we turn off the rotating magnetic field and leave the atoms to precess in  $B_0$ . This gives us at later times

$$\begin{aligned}v_{x\Delta}(t) &= \cos(\Delta(t - t_3))v_{x\Delta}(t_3) - \sin(\Delta(t - t_3))v_{y\Delta}(t_3) \\&= \cos(\Delta(t - t_3))\sin(\Delta(t_2 - t_1)) - \sin(\Delta(t - t_3))\cos(\Delta(t_2 - t_1)) \\&= -\sin(\Delta(t + t_1 - 2t_3)), \\v_{y\Delta}(t) &= \cos(\Delta(t - t_3))v_{y\Delta}(t_3) + \sin(\Delta(t - t_3))v_{x\Delta}(t_3) \\&= \cos(\Delta(t - t_3))\cos(\Delta(t_3 - t_1)) + \sin(\Delta(t - t_3))\sin(\Delta(t_3 - t_1)) \\&= \cos(\Delta(t + t_1 - 2t_3)), \\v_{z\Delta}(t) &= 0.\end{aligned}$$

Already we can see that at  $t_4 = 2t_3 - t_1$   $v_{x\Delta} = 0 = v_{z\Delta}$  and  $v_{y\Delta} = 1$  independent of  $\Delta$ .

Note that for times  $t > t_3$   $v_{x\Delta} + iv_{y\Delta} = i \exp(i\Delta(t + t_1 - 2t_3))$  and this allows us to perform exactly the same average as before

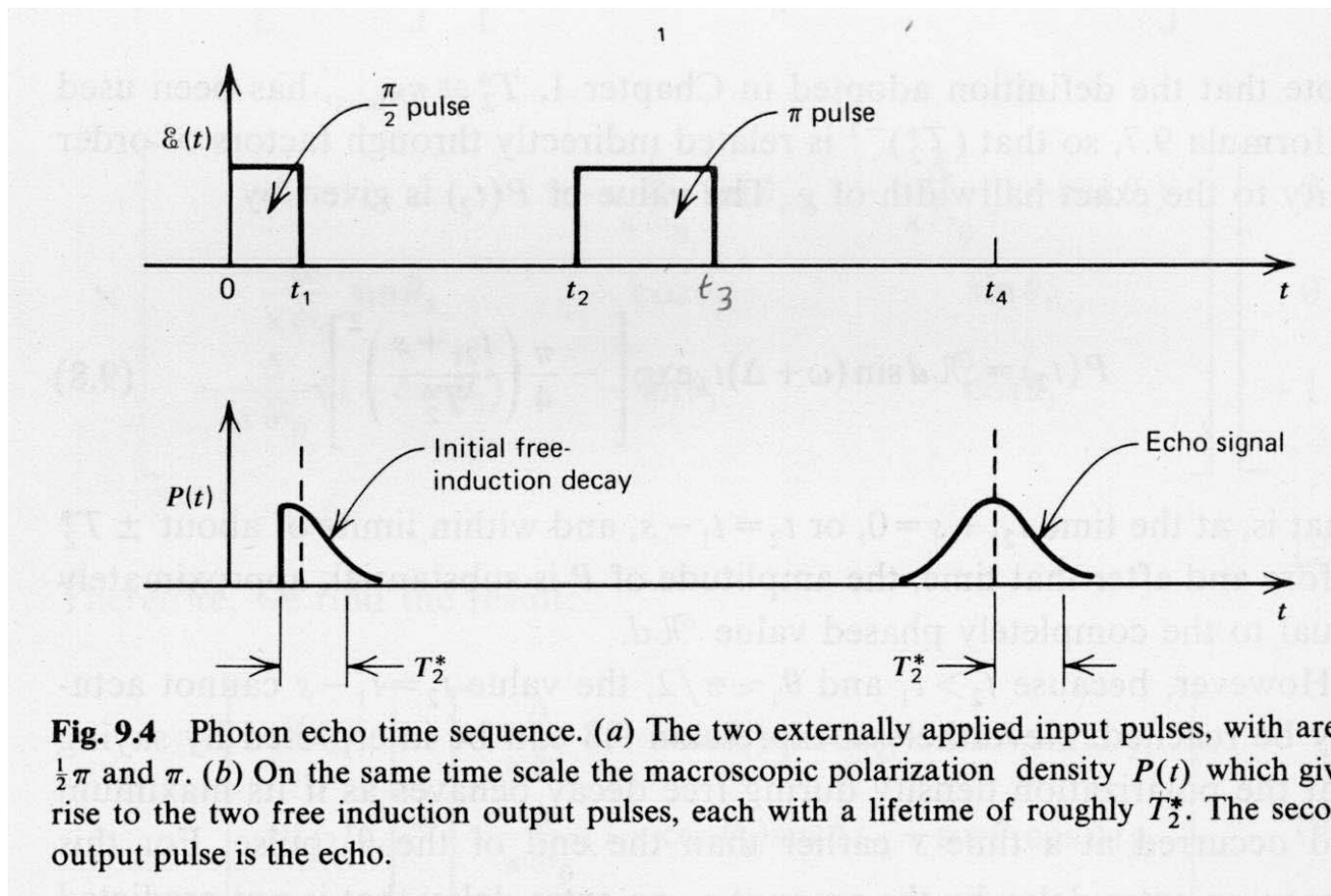
$$\begin{aligned}v_x + iv_y &= \frac{T_2^*}{\pi} \int_{-\infty}^{\infty} \exp(-\Delta^2 T_2^{*2}/\pi)(v_{x\Delta} + iv_{y\Delta})d\Delta. \\&= i \frac{T_2^*}{\pi} \int_{-\infty}^{\infty} \exp(-\Delta^2 T_2^{*2}/\pi)(i\Delta(t + t_1 - 2t_3))d\Delta.\end{aligned}$$

Performing the integral tells us that the Bloch vector has a Gaussian pulse of width about  $T_2^*$  around time  $t_4$

$$v_x + iv_y = i \exp\left[-\frac{\pi}{4} \left(\frac{t + t_1 - 2t_3}{T_2^*}\right)^2\right].$$

This pulse of bulk magnetization is known as the spin echo. It could be detected by looking at its induced current in current coils exactly as discussed above for free induction decay.

Just to keep each these different times straight here is a schematic, again from Allen and Eberly, of the spin echo process.



**Fig. 9.4** Photon echo time sequence. (a) The two externally applied input pulses, with are  $\frac{1}{2}\pi$  and  $\pi$ . (b) On the same time scale the macroscopic polarization density  $P(t)$  which give rise to the two free induction output pulses, each with a lifetime of roughly  $T_2^*$ . The second output pulse is the echo.

In practice the echo will not be as intense as the original signal. This may be due to two important causes. Firstly in a liquid atoms may simple move around and this will lead to later echos not being as strong as earlier ones. Perhaps more importantly though subsequent error signals will decay exponentially due to the intrinsic dephasing  $T_2$  and this provides a useful method of estimating  $T_2$ . It is also possible to perform modified echo experiments which determine  $T_1$ .

## Molecular Beam Experiments

(Ramsey, Molecular Beams Ch 1, 5)

Now we will shift focus to a slightly different experimental arrangement for observing magnetic resonance effects. In molecular beam experiments a beam of a particular atom or

molecule emerges from an oven and passes through a spatially and temporally varying magnetic field. Typically there is a large background field  $B_0$  as before with some oscillating or rotating component  $b_1$  at a frequency  $\omega$ . Effects like the  $\pi$ -pulses we discussed earlier can be achieved by confining the  $b_1$  field to a region of length  $l$ . The atoms will then see a pulse of length  $t = l/v$  where  $v$  is the velocity of the atom. Typically the magnetic dipoles of interest are again nuclear magnetic moments. After the region of the beam that corresponds to the pulse sequence in an NMR experiment there will be a spatially varying magnetic field designed to measure the orientation of the magnetic dipoles by measuring the deflection of the atoms as in a Stern-Gerlach experiment. For example it is possible to measure the population of the  $|+z\rangle$  and  $|-z\rangle$  levels in this fashion since they will be deflected in opposite directions with an appropriate magnetic field. At the end of the beam atoms are detected by, for example, having the beam hit a heated wire filament which will cause the atoms to ionise and a current to flow in the wire.

Once again the simplest thing we might wish to measure is the Larmor frequency  $\omega_L$  of the magnetic moment due to the field  $B_0$ . Rabi's original method of doing this is to make use of magnetic resonance by detecting the probability for the atomic spin to flip from  $|-z\rangle$  to  $|+z\rangle$  as the frequency  $\omega$  of the field  $b_1$  is tuned.

There are two main inhomogeneities in these experiments that lead to similar effects to the free induction decay and spin echos which we considered in the previous sections. Firstly each of the atoms will have a slightly different velocity and will therefore spend a different amount of time the  $b_1$  field. Secondly the field  $B_0$  will not be completely constant over the entire length of the beam. Both of these effects tend to mean that atoms are likely to undergo spin flips for a range of frequencies  $\omega$  close to  $\omega_L$ . This makes it more difficult to determine the exact Larmor frequency. Ramsey's method separated oscillatory fields is one way of achieving a narrower resonance and thus getting a more accurate measurement.

## Ramsey Fringes

The pulse sequence for the Ramsey method of separated oscillatory fields is very similar to the pulses used in a spin echo experiment. In the interests of brevity I will give a simple discussion rather than do all the math.

The first region that the beam passes through has a background magnetic field  $B_0$  and an oscillating field  $b_1$  at frequency  $\omega$ . The atoms start of in the state

$$\rho(l) = |-z\rangle\langle -z|.$$

This first region is sufficiently long that if the oscillating field is on resonance it performs a  $\pi/2$ -pulse preparing the atoms in the state

$$\rho(l) = |-y\rangle\langle -y|.$$

The atoms then move through a region of length  $L$  with just the magnetic field  $B_0$ . Thus the atoms precess at the frequency  $\bar{\omega}_L$  set by the spatially averaged magnetic field  $\bar{B}_0$ . In a frame rotating at frequency  $\omega$  the state is

$$v_x(l+L, v) + iv_y(l+L, v) = -i \exp[i(\bar{\omega}_L - \omega)L/v]$$

at  $l+L$  when the atom passes into the second oscillatory field which is also assumed to be oscillating at  $\omega$ . Suppose that  $\omega = \omega_L$  then

$$\rho(l+L) = |-y\rangle\langle -y|$$

regardless of the atomic velocity. The second oscillating field is also arranged to be a  $\pi/2$ -pulse if it is on resonance so that now we have

$$\rho(2l+L) = |+z\rangle\langle +z|$$

and so the atoms have undergone a spin flip.

If on the other hand the oscillating field is not on resonance the atom will not end up in  $|+z\rangle$ . In this case the Bloch vector will precess around the  $z$ -axis even in the rotating frame. The longer the distance between the two oscillating fields the larger the rotation. In fact in the off resonance situation the atom can rotate around the  $z$ -axis such that

$$\rho'(l+L) = |+y\rangle\langle +y|$$

and then the second  $\pi/2$ -pulse returns the atom to  $|-z\rangle$ ! So in the off resonance situation there is some tendency for the atom to returned to the original state by the second pulse. This contributes to the narrow linewidth of the Ramsey technique.

What is not obvious from this is what happens when we average over atomic velocities or consider arbitrary detunings during the  $\pi/2$ -pulses. For details see the book by Ramsey. It turns out that the width of the transition may be made small by making the distance between the two oscillating fields as large as possible. In fact

$$\Delta\omega \simeq 0.65\alpha/L$$

where  $\alpha$  is the most likely atomic velocity. (The constant in front depends on the velocity distribution but this is well understood for atomic beams). This is narrower than the width of the transition for a direct magnetic resonance experiment even if we don't include variations of  $B_0$ . Below is a plot of the excited state population for a realistic set of parameters taken from Ramsey's book.

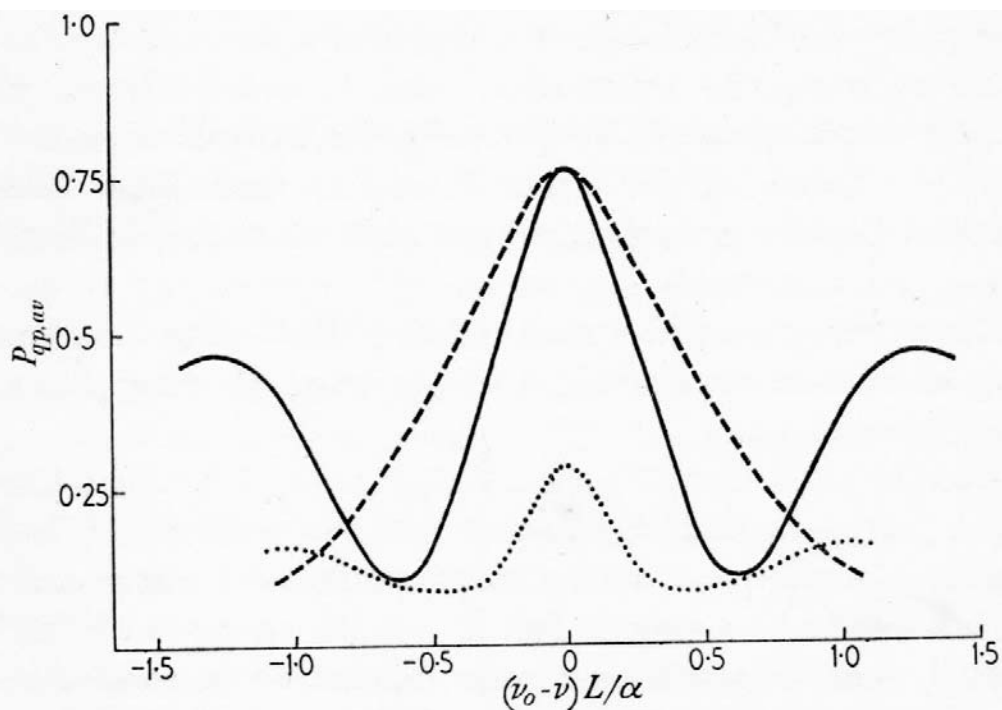


FIG. V. 2. Transition probability averaged over molecular velocity as a function of frequency near resonance.  $L$  = distance between oscillating field regions,  $\alpha$  = most probable molecule velocity, and  $\nu$  = oscillator frequency. The full curve is the transition probability for the optimum perturbation ( $2bl/\alpha = 0.600\pi$ ). The dotted curve is for one-third the oscillating field. The dashed curve is for the Rabi method with the same length of  $C$  field (RAM  $50c$ ).