Ph195a lecture notes, 11/14/01

Today's lecture follows Cohen-Tannoudji, Diu, and Laloe, Chapter IV.

[Cohen-Tannoudji *et al*, Ch. IV Figure 1]

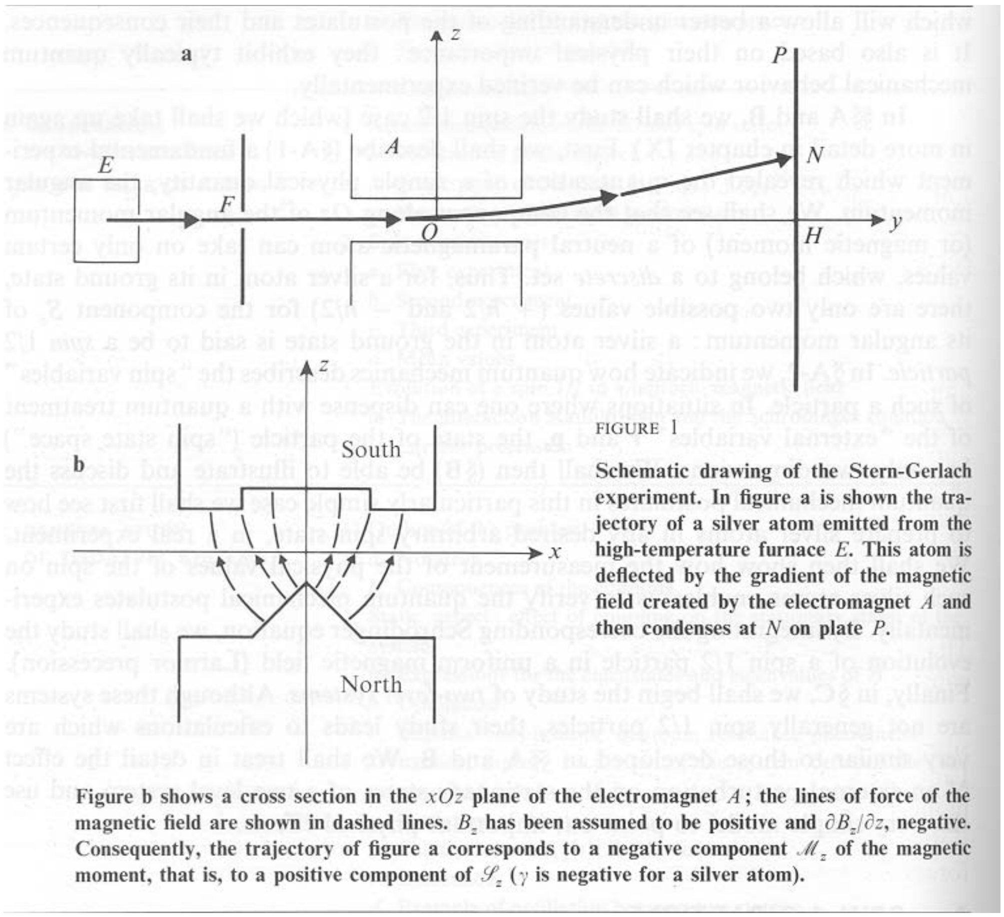


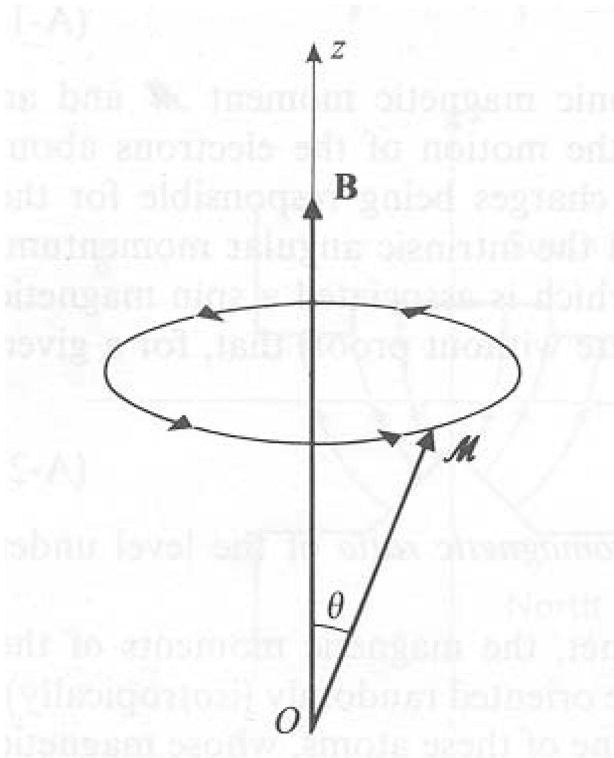
FIGURE 1

Schematic drawing of the Stern-Gerlach experiment. In figure a is shown the trajectory of a silver atom emitted from the high-temperature furnace *E*. This atom is deflected by the gradient of the magnetic field created by the electromagnet *A* and then condenses at *N* on plate *P*.

Figure b shows a cross section in the xOz plane of the electromagnet A; the lines of force of the magnetic field are shown in dashed lines. B_z has been assumed to be positive and $\partial B_z/\partial z$, negative. Consequently, the trajectory of figure a corresponds to a negative component $\mathscr{M}_z$ of the magnetic moment, that is, to a positive component of $\mathscr{S}_z$ (γ is negative for a silver atom).

$$
\begin{aligned}
W &= -\vec{\mu} \cdot \vec{B} \\
\vec{F} &= -\nabla W \\
&= \nabla\left(\vec{\mu} \cdot \vec{B}\right) \\
\vec{\mu} &= \gamma \vec{S} \\
\vec{\Gamma} &= \vec{\mu} \times \vec{B}
\end{aligned}
$$

[Cohen-Tannoudji *et al*, Ch. IV Figure 2]



Due to precessional averaging of transverse components,

$$\vec{F} = \nabla(\mu_z B_z).$$

Quantum description:

$$|\Psi\rangle \in H_2$$
$$H_2 = \text{span}\{|+\rangle, |-\rangle\}$$
$$\langle +|+\rangle = \langle -|-\rangle = 1,$$
$$\langle +|-\rangle = 0.$$

$$\mathbf{S}_z = +\frac{\hbar}{2}|+\rangle\langle+| - \frac{\hbar}{2}|-\rangle\langle-|.$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\mathbf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$= +\frac{\hbar}{2}|+_x\rangle\langle+_x| - \frac{\hbar}{2}|-_x\rangle\langle-_x|$$

$$|+_x\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle),$$

$$|-_x\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle),$$

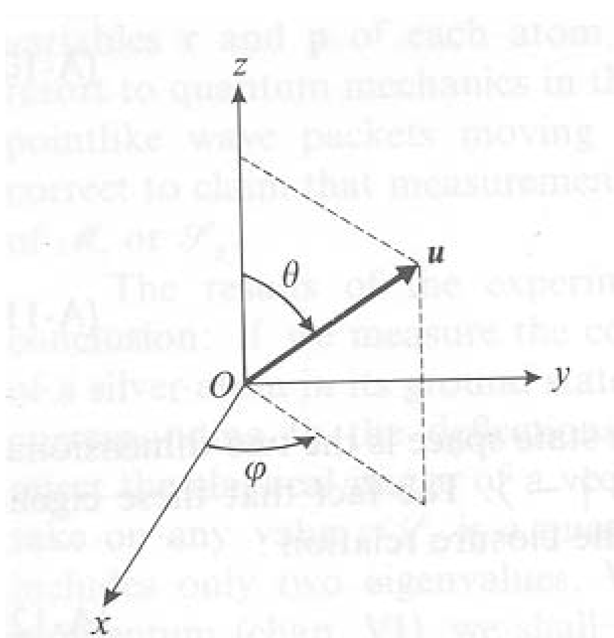
$$\mathbf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$= +\frac{\hbar}{2}|+_y\rangle\langle+_y| - \frac{\hbar}{2}|-_y\rangle\langle-_y|$$

$$|+_y\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle),$$

$$|-_x\rangle = \frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle).$$

[Cohen-Tannoudji *et al*, Ch. IV Figure 4]



$$
\begin{aligned}
\mathbf{S}_u &= \mathbf{S}_x \sin\theta\cos\phi + \mathbf{S}_y \sin\theta\sin\phi + \mathbf{S}_z\cos\theta \\
&= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\exp(-i\phi) \\ \sin\theta\exp(+i\phi) & -\cos\theta \end{pmatrix}, \\
|+_u\rangle &= \cos\frac{\theta}{2}\exp(-i\phi/2)|+\rangle + \sin\frac{\theta}{2}\exp(+i\phi/2)|-\rangle, \\
|-_u\rangle &= -\sin\frac{\theta}{2}\exp(-i\phi/2)|+\rangle + \cos\frac{\theta}{2}\exp(+i\phi/2)|-\rangle.
\end{aligned}
$$

Dynamics in the quantum theory:

$$
\begin{aligned}
W &= -\vec{\mu}\bullet\vec{B}, \\
\vec{\mu} &= \gamma\vec{S}, \\
\mathbf{H} &= -\vec{\boldsymbol{\mu}}\bullet\vec{B} \\
\vec{\boldsymbol{\mu}} &= \gamma(\mathbf{S}_x\hat{x} + \mathbf{S}_y\bullet + \mathbf{S}_z\hat{z}), \\
&\equiv \gamma\vec{\mathbf{S}} \\
\mathbf{H} &= -\gamma(\mathbf{S}_x B_x + \mathbf{S}_y B_y + \mathbf{S}_z B_z).
\end{aligned}
$$

Let’s think about $\vec{B} = B_0\hat{z}$. Then

$$
\begin{aligned}
\mathbf{H} &= -\gamma B_0\mathbf{S}_z \\
&= \omega_0\mathbf{S}_z, \\
\omega_0 &\equiv -\gamma B_0,
\end{aligned}
$$

where ω_0 is called the “Larmor frequency.”
Think about SE:

$$
i\hbar\frac{d}{dt}|\Psi\rangle = \mathbf{H}|\Psi\rangle
$$

with $|\Psi(0)\rangle = |+\rangle$. Then

$$
\begin{aligned}
i\hbar\frac{d}{dt}|\Psi\rangle &= \frac{\hbar\omega_0}{2}(|+\rangle\langle+| - |-\rangle\langle-|)|\Psi\rangle \\
\frac{d}{dt}|+\rangle &= \frac{-i\omega_0}{2}|+\rangle \\
|\Psi(t)\rangle &= \exp(-i\omega_0 t/2)|+\rangle.
\end{aligned}
$$

What about $|\Psi(0)\rangle = |+_x\rangle$?

$$
\begin{aligned}
|+_x\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \\
|\Psi(0)\rangle &= |+_x\rangle \\
|\Psi(t)\rangle &= \frac{1}{\sqrt{2}}(\exp(-i\omega_0 t/2)|+\rangle + \exp(i\omega_0 t/2)|-\rangle).
\end{aligned}
$$

Recalling the general expression:

$$
|+_u\rangle = \cos\frac{\theta}{2}\exp(-i\phi/2)|+\rangle + \sin\frac{\theta}{2}\exp(+i\phi/2)|-\rangle,
$$

Apparently $\theta = \pi/2$ always, and $\phi = \omega_0 t$.
And for $|\Psi(0)\rangle = |+_u\rangle = \cos\frac{\theta}{2}\exp(-i\phi/2)|+\rangle + \sin\frac{\theta}{2}\exp(+i\phi/2)|-\rangle$, for some arbitrary θ and

ϕ,

$$|\Psi(t)\rangle = \cos\frac{\theta}{2}\exp(-i\phi/2)\exp(-i\omega_0 t/2)|+\rangle + \sin\frac{\theta}{2}\exp(+i\phi/2)\exp(i\omega_0 t/2)|-\rangle$$
$$= \cos\frac{\theta}{2}\exp(-i(\phi+\omega_0 t)/2)|+\rangle + \sin\frac{\theta}{2}\exp(+i(\phi+\omega_0 t)/2)|-\rangle,$$

again we see θ is constant, ϕ increases linearly with t.

Let's now consider

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{W},$$

where

$$\mathbf{H}_0 = \omega_0 \mathbf{S}_z = \frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we let **W** be something completely arbitrary,

$$\mathbf{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

in the z basis. Then the total Hamiltonian is

$$\mathbf{H} = \begin{pmatrix} E_1 + W_{11} & W_{12} \\ W_{21} & E_2 + W_{22} \end{pmatrix},$$
$$E_1 = \hbar\omega_0/2, \quad E_2 = -\hbar\omega_0/2,$$

which can analytically be solved to find

$$E_+ = \frac{1}{2}(E_1 + W_{11} + E_2 + W_{22}) + \frac{1}{2}\sqrt{(E_1 + W_{11} - E_2 - W_{22})^2 + 4|W_{12}|^2},$$
$$E_+ = \frac{1}{2}(E_1 + W_{11} + E_2 + W_{22}) - \frac{1}{2}\sqrt{(E_1 + W_{11} - E_2 - W_{22})^2 + 4|W_{12}|^2}.$$

If we simplify to $W_{11} = W_{22} = 0$, we find

$$E_\pm = E_m \pm \sqrt{\Delta^2 + |W_{12}|^2},$$
$$E_m \equiv \frac{1}{2}(E_1 + E_2),$$
$$\Delta \equiv \frac{1}{2}(E_1 - E_2).$$

Assuming $|W_{12}| \ll \Delta$ (which for us would be $B_x \ll B_z$), then

$$E_\pm = E_m \pm \Delta\left(1 + \frac{1}{2}\left(\frac{|W_{12}|}{\Delta}\right)^2 + \cdots\right).$$

Generally gives "anti-crossing" diagram [Cohen-Tannoudji *et al*, Ch. IV Figure 11]:

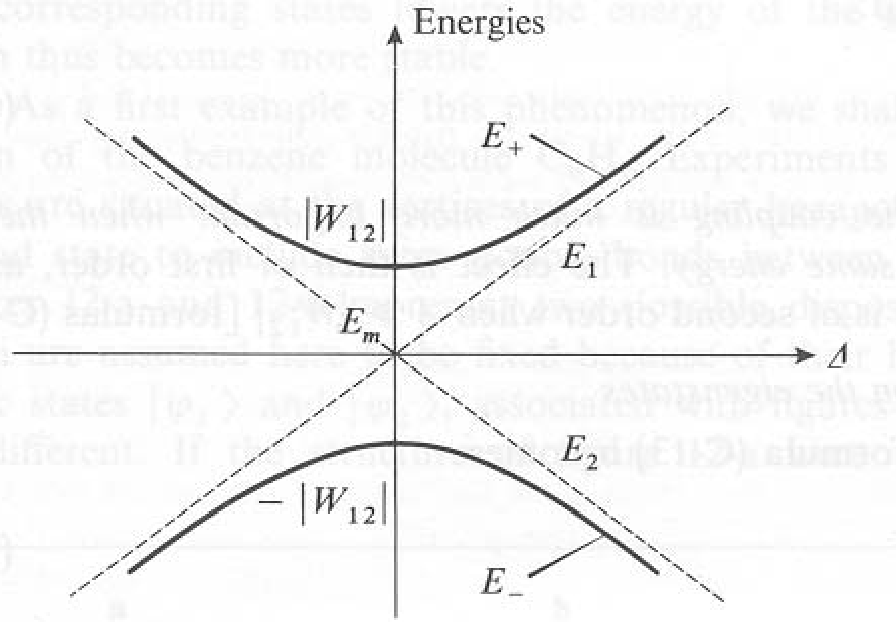


FIGURE 11