

Fighting decoherence in a continuous two-qubit odd/even parity measurement with a closed loop setup

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A parity measurement on two qubits, each consisting of a single atom in a cavity, can be realized by measuring the phase shift of a probe beam, which interacts sequentially with the two qubits, but imperfections lead to decoherence within the subspaces of a given parity. We demonstrate that a different setup, where the probe light interacts repeatedly with the qubits, can reduce the rate of decoherence within the odd or the even parity subspace significantly. We consider both the case of a resonant and the case of a nonresonant light-atom interaction and find that the performance is comparable if the parameters are chosen appropriately.

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I. INTRODUCTION

It has been realized that measurements provide a powerful toolbox to manipulate the state of quantum systems in ways that might be difficult to achieve with standard interactions. An example is the possibility of quantum computing with only single photon sources, linear optics, and photo detectors [1]. A measurement may consist in observing a part of a larger system such as in several recent experiments preparing various quantum states of light by conditional detection of a small part of the light field [2–4]. Alternatively, one may use a probe to gain information about the state of a system as in atomic spin squeezing by use of Faraday rotation [5]. In the latter case, the measurement should ideally be constructed in such a way that the interaction between the probe and the system does not affect the time evolution of the measured observable.

A particularly interesting case arises, when the measured observable has degenerate eigenvalues, since one may then obtain partial information about the state of the system, while leaving a specific subspace unaffected. This opens the way to use measurements to prepare macroscopic quantum superposition states [6, 7] and to detect errors in quantum computations without destroying the quantum information stored in the qubits [8]. An example is the three-qubit bit-flip code, where each qubit is encoded in the two states $|000\rangle$ and $|111\rangle$ of three physical qubits, and a single bit-flip error can be detected by measuring the parity of qubits 1 and 2 and the parity of qubits 2 and 3. Other applications of parity measurements include entanglement generation [9], quantum computing [10], and state transfer [11].

Recently, it has been proposed to perform a continuous parity measurement on two qubits in a cavity quantum electrodynamics network by allowing a coherent state probe to interact sequentially with the two qubits and then measure the phase shift imposed on the probe light with a homodyne detector [12]. More precisely, each

qubit is encoded in two ground state levels $|0\rangle$ and $|1\rangle$ of a single atom in a cavity, and the probe light couples the state $|1\rangle$ resonantly to an excited state $|e\rangle$. In the strong coupling limit, the vacuum Rabi splitting prevents the light from entering into the cavity if the atom is in the state $|1\rangle$ [13], and each interaction thus leaves the probe light unaffected except for a conditional phase shift of π [14]. The measurement takes a finite amount of time due to uncertainties arising from shot noise, and it is suitable for continuous quantum error correction, where feedback is used to keep the state of the qubits within the code space as far as possible [15, 16]. Quantum error correction is often thought of as strong measurements followed by fast corrections, but continuous error correction contains some interesting aspects due to the intrinsic robustness of feedback control. In fact, in [17] a setup has been proposed in which the above interaction is used, but rather than detecting the probe field, the probe field is used in a coherent feedback loop, which automatically corrects errors such that the state of the qubits remains close to the code space at all times.

Independent of whether the probe field is detected or used for coherent feedback, it is important to minimize decoherence within the parity subspaces due to the interaction with the probe. The above scheme is sensitive to light field losses between and within the two cavities as well as to spontaneous emission from the excited state of each atom, arising because the field is not completely expelled when the coupling strength is finite. In the present paper we show that a slight change of the setup can decrease the rate of decoherence in the odd or the even parity subspace by about one order of magnitude. The basic idea is to allow the light field to interact several times with the two qubits by reflecting some of the light emerging from the second cavity back onto the input mirror of the first cavity as shown in Fig. 1. For the odd subspace (and $P = 1$), the light field experiences a phase shift of π for each round trip in the loop, and the resulting destructive interference between light fields having traveled a different number of round trips in the

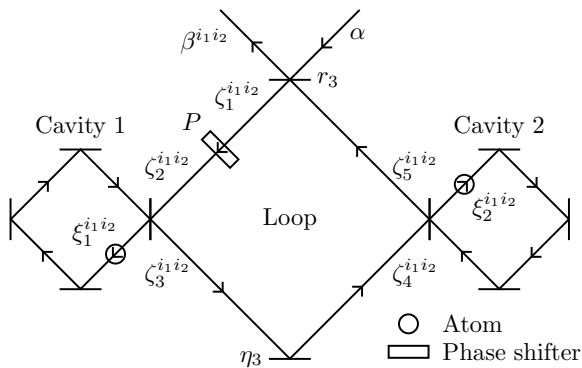


FIG. 1: Proposed setup to perform a parity measurement on two qubits, each encoded in two ground state levels of a single atom in a cavity. The input field α is in a continuous coherent state, and the output field β is observed with a homodyne detector. The Greek letters label the field at different positions, r_3^2 is the reflectivity of the input beam splitter, $P = e^{i\psi}$ is the phase factor imposed on the light field by the phase shifter, and $1 - \eta_3$ is the fraction of photons that are lost, when light travels from cavity 1 to cavity 2.

loop almost extinguishes the field. The difference between the field amplitudes obtained for the two possible states within the odd subspace is thus small, and this improves the robustness of the measurement with respect to loss. The destructive interference also decreases the field amplitudes inside the qubit cavities, which reduces the rate of spontaneous emission events and further decreases the decoherence rate. On the other hand, the sensitivity of the measurement is not decreased, since the even parity subspace leads to constructive interference in the loop, i.e., the phase of the output field differs by π for the two subspaces as before. We note that the roles of the odd and the even parity subspaces could be exchanged by adding a phase shift of π within the loop, but in the following we focus on protecting the odd subspace. The reason for this is that the output field is not exactly the same for the two even parity states, because spontaneous emission is possible for $|11\rangle$ but not for $|00\rangle$, and as a result the homodyne detection of the output field does not leave the even subspace completely unaffected. While this makes the analysis slightly more complicated, it will not have any significant effects, since the corrections are of second order. A parity measurement can also be realized with a light field, which is far detuned from the atomic transition, and this possibility will also be investigated.

In Sec. II we provide the theoretical framework required to analyze the proposed setup, and we obtain expressions for the field amplitudes, which substantiate the above statements. In Sec. III we derive an equation for the time evolution of the purity of a state restricted to either the odd or the even parity subspace and identify the parameters, which characterize the performance of the proposal. These parameters are evaluated for a nonresonant light-atom interaction in Sec. IV and for a

resonant light-atom interaction in Sec. V. We also compute the optimal value of the reflectivity r_3^2 of the input beam splitter and compare the performance of the resonant and the nonresonant interaction. Finally, Sec. VI concludes the paper.

II. SYSTEM STATE AND PARITY MEASUREMENTS

A. General form of the system density operator

To analyze the setup in Fig. 1, we first consider the interaction between an atom and a single light field mode. The atom is assumed to have two degenerate ground state levels $|0\rangle$ and $|1\rangle$, and the light field couples the state $|1\rangle$ to an excited state $|e\rangle$. In a frame rotating with the angular frequency ω of the light field, the master equation for the interaction takes the form

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \frac{\Gamma}{2}(2\sigma\rho\sigma^\dagger - \sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma), \quad (1)$$

where

$$H = \hbar g(\hat{a}\sigma^\dagger + \hat{a}^\dagger\sigma) + \hbar\Delta\sigma^\dagger\sigma \quad (2)$$

is the Hamiltonian, ρ is the density operator, Γ is the decay rate of the excited state of the atom, $\sigma = |1\rangle\langle e|$ is the atomic lowering operator, g is the light-atom coupling strength, \hat{a} is the field annihilation operator, and $\Delta = \omega_{\text{at}} - \omega$ is the detuning between the atomic resonance angular frequency and the angular frequency of the light field.

In the Markov limit, where all time delays go to zero, one could derive a quantum filtering equation for the time evolution of the state of the full system depicted in Fig. 1, for instance by using the time evolution obtained for the measurement free case in [18] and add homodyne detection [19]. Here, however, we are solely interested in the parameter regime, where the probability to populate the excited atomic level is small since observation of a photon emitted by spontaneous emission allows us to distinguish the two states in the odd subspace and to distinguish the two states in the even subspace. We thus assume that either the decay rate or the absolute value of the light-atom detuning is large compared to the effective driving of the atomic transition, which sum up to the condition

$$g^2|\langle\hat{a}\rangle|^2 \ll \Gamma^2/4 + \Delta^2. \quad (3)$$

When (3) is satisfied, we can use the procedure outlined in [20, 21] to obtain an approximation to the light-atom interaction dynamics by assuming $g\hat{a} \propto k$, $\Gamma \propto k^2$, $\Delta \propto k^2$ and taking the limit $k \rightarrow \infty$, which transforms (1) into

$$\frac{d\rho}{dt} = \frac{i\Delta g^2}{(\Gamma/2)^2 + \Delta^2} [\hat{a}^\dagger\hat{a}|1\rangle\langle 1|, \rho] + \frac{\Gamma}{2} \frac{g^2}{(\Gamma/2)^2 + \Delta^2} \times (2\hat{a}|1\rangle\langle 1|\rho|1\rangle\langle 1|\hat{a}^\dagger - \hat{a}^\dagger\hat{a}|1\rangle\langle 1|\rho - \rho|1\rangle\langle 1|\hat{a}^\dagger\hat{a}), \quad (4)$$

where the Hilbert space of the atom is now restricted to $\text{span}\{|0\rangle, |1\rangle\}$. The interaction in (4) is a state dependent phase shifter combined with a state dependent loss term, which can be modeled as a beam splitter coupling the cavity field to a vacuum field followed by a partial trace over the vacuum field. Since beam splitters, mirrors, and phase shifters all transform the field operators linearly, and since homodyne detection of the output field does not affect the coherent amplitudes of the unobserved field modes, the state of the system at time t may be represented as

$$\rho(t) = \sum_{i_1, j_1, i_2, j_2=0}^1 c_{i_1 j_1 i_2 j_2}(t) \bigotimes_{k=1}^N |\zeta_k^{i_1 i_2} \sqrt{T_k}\rangle \langle \zeta_k^{j_1 j_2} \sqrt{T_k}| \bigotimes_{q=1}^2 |\xi_q^{i_1 i_2} \sqrt{\tau_q}\rangle \langle \xi_q^{j_1 j_2} \sqrt{\tau_q}| \otimes |i_1\rangle \langle j_1| \otimes |i_2\rangle \langle j_2|. \quad (5)$$

Here, i_q and j_q specify the state of the atom in cavity q , and $c_{i_1 j_1 i_2 j_2}(t)$ are time dependent scalars. The light field in the loop is divided into N modes of temporal width T_k and conditional amplitude $\zeta_k^{i_1 i_2}$ normalized such that $|\zeta_k^{i_1 i_2}|^2 T_k$ is the conditional expectation value of the number of photons in the k th mode. In Fig. 1, we have labeled five such modes. The fields in cavity 1 and 2 are treated as single mode fields with conditional amplitudes $\xi_q^{i_1 i_2}$, and τ_q is the round trip time of light in cavity q .

B. Steady state field amplitudes

In general, the conditional field amplitudes are time dependent, but if the input field is turned on at time $t = 0$ and is constant for $t > 0$, the amplitudes approach constant values at a rate which, in the limit $r_3 \rightarrow 0$, is the smallest of the decay rates of the light fields in cavity 1 and 2. To compute the steady state field amplitudes, we note that the light-atom interaction in cavity $q = 1, 2$ combined with the cavity-light detuning transform the cavity field amplitude as $\xi_q^{i_1 i_2} \rightarrow f_q^{i_1 i_2} \xi_q^{i_1 i_2}$, where

$$f_q^{i_1 i_2} = \exp\left(-\left(\frac{\Gamma_q}{2} - i\Delta_q\right) \frac{g_q^2 \delta_{1i_q}}{\Gamma_q^2/4 + \Delta_q^2} \tau_q - i\delta_q \tau_q\right), \quad (6)$$

δ_{ij} is the Kronecker delta, and $\delta_q = \omega_{c,q} - \omega$ is the detuning between the resonance angular frequency of cavity q and the angular frequency of the light field. Treating the mirrors as beam splitters with zero transmissivity and assuming that a fraction $1 - \eta_3$ of the photons is lost in the reflection at the lower mirror in Fig. 1, we then obtain $\zeta_1^{i_1 i_2} = t_3 \alpha + i r_3 \zeta_5^{i_1 i_2}$, $\zeta_2^{i_1 i_2} = P \zeta_1^{i_1 i_2}$, $\zeta_3^{i_1 i_2} = -i F_1^{i_1} \zeta_2^{i_1 i_2}$, $\zeta_4^{i_1 i_2} = i \sqrt{\eta_3} \zeta_3^{i_1 i_2}$, $\zeta_5^{i_1 i_2} = -i F_2^{i_2} \zeta_4^{i_1 i_2}$, $\beta^{i_1 i_2} = t_3 \zeta_5^{i_1 i_2} + i r_3 \alpha$, $\xi_1^{i_1 i_2} = t_1 \zeta_2^{i_1 i_2} / (1 - r_1 f_1^{i_1})$, and $\xi_2^{i_1 i_2} = t_2 \zeta_4^{i_1 i_2} / (1 - r_2 f_2^{i_2})$, where $F_q^{i_1 i_2} = (f_q^{i_1 i_2} - r_q) / (1 - r_q f_q^{i_1 i_2})$ and $r_q^2 = 1 - t_q^2$ is the reflectivity of the input beam

splitter of cavity q . From these relations we derive

$$\zeta_2^{i_1 i_2} = \frac{P t_3}{1 - r_3 P F_1^{i_1} F_2^{i_2} \sqrt{\eta_3}} \alpha, \quad (7)$$

$$\beta^{i_1 i_2} = i \frac{r_3 - P F_1^{i_1} F_2^{i_2} \sqrt{\eta_3}}{1 - r_3 P F_1^{i_1} F_2^{i_2} \sqrt{\eta_3}} \alpha. \quad (8)$$

If the atom and cavity parameters are the same for cavity 1 and 2, it follows that $\beta^{i_1 i_2} = \beta^{i_2 i_1}$, i.e., the output field is exactly the same for the two two-qubit states with odd parity. This is a very important property of the system, since it ensures that it is impossible to destroy a superposition of odd parity states by doing any kind of measurements on the output field, and, in particular, inefficient detection does not give rise to decoherence in the odd subspace. In appendix A, we show that this is valid even during the transient and for a time dependent input field.

Since $f_q^{i_1 i_2}$ is typically very close to unity, we should choose r_q^2 , $q = 1, 2$, to be close to unity as well in order to allow $F_q^{i_1 i_2}$ to be significantly different from unity. We can thus expand $f_q^{i_1 i_2}$ and r_q to first order to obtain

$$\xi_1^{i_1 i_2} = \frac{2}{\sqrt{\kappa_1 \tau_1}} \frac{(1 + D_1^2) \zeta_2^{i_1 i_2}}{1 + D_1^2 + 2C_1 \delta_{1i_1} - 2iC_1 D_1 (\delta_{1i_1} - 1/2)} \quad (9)$$

and

$$F_q^{i_1 i_2} = \frac{1 + D_q^2 - 2C_q \delta_{1i_q} + 2iD_q C_q (\delta_{1i_q} - 1/2)}{1 + D_q^2 + 2C_q \delta_{1i_q} - 2iD_q C_q (\delta_{1i_q} - 1/2)}, \quad (10)$$

where $C_q = 2g_q^2 / (\kappa_q \Gamma_q)$ is the cooperativity parameter, $\kappa_q = t_q^2 / \tau_q$ is the cavity decay rate, $D_q = 2\Delta_q / \Gamma_q$, and for reasons that will appear below, we have chosen the cavity-light detuning such that $2\delta_q / \kappa_q = D_q C_q / (1 + D_q^2)$. Note that $|F_q^{i_1 i_2}| \leq 1$, which is a consequence of energy conservation.

For a resonant light-atom interaction $D_q = 0$, we observe that $F_q^0 = 1$, while $F_q^1 \rightarrow -1$ for $C_q \rightarrow \infty$, i.e., the light experiences an additional phase shift of π if the atom is in the state $|1\rangle$ rather than $|0\rangle$. As noted in the Introduction, we obtain destructive interference in the loop if we choose $P = 1$, and for $\eta_3 = 1$ we find $\beta^{00} = -i\alpha$, $\beta^{10} = \beta^{01} \rightarrow i\alpha$, and $\beta^{11} \rightarrow -i\alpha$, which is the optimal situation for a parity measurement. Note also that $\zeta_2^{10} = \zeta_2^{01}$ is small when r_3^2 is high, and this decreases the number of photons lost through spontaneous emission and other light field losses. In the nonresonant case $D_q \sim C_q \gg 1$, $F_q^1 \approx (F_q^0)^*$ due to the specific choice of cavity-light detuning above. The light field thus experiences one phase shift if the atom is in state $|1\rangle$ and the opposite phase shift if the atom is in state $|0\rangle$. The overall phase shift per round trip is thus zero in the odd subspace, and we choose $P = -1$ to obtain destructive interference in the loop. For $\alpha = \alpha^*$ and identical atom and cavity parameters, the conditional output field amplitudes satisfy $\beta^{10} = \beta^{01}$ and $\beta^{11} = -(\beta^{00})^*$, and a

parity measurement can be achieved by detecting the p -quadrature of the output field. If the atom and cavity parameters are not identical and $F_1^i \neq F_2^i$, $i = 0, 1$, we still have $\beta^{10} = -(\beta^{01})^*$ and $\beta^{11} = -(\beta^{00})^*$ and this is sufficient to facilitate a parity measurement, but detection inefficiency now leads to decoherence within the odd subspace. The relations $\beta^{10} = -(\beta^{01})^*$ and $\beta^{11} = -(\beta^{00})^*$ are also valid during the transient as may be inferred from Eq. (A1).

C. Validity of the approximation

Having computed the steady state field amplitudes, we can now check whether assumption (3) is consistent. The assumption is only relevant if the considered atom interacts with the light field, and for cavity 1 we thus compute $|\langle \hat{a} \rangle|^2$ conditioned on atom 1 being in state $|1\rangle$ and atom 2 being in state $|i_2\rangle$, which leads to the condition

$$\frac{2|\alpha|^2}{\Gamma} \times \frac{4C_1(1 + D_1^2)}{(1 + D_1^2 + 2C_1)^2 + C_1^2 D_1^2} \times \frac{t_3^2}{|1 - r_3 P F_1^1 F_2^{i_2} \sqrt{\eta_3}|^2} \ll 1. \quad (11)$$

The first factor is the number of photons in the input beam per unit time relative to the average spontaneous emission rate of an atom with an average probability of one half to be in the excited state. We anticipate that $|\alpha|^2$ is chosen such that this factor is of order unity or smaller. The second factor is small if $C_1 \gg 1$. In fact, this factor is of order $1/C_1$ independent of whether $D_1 \sim 0$, $D_1 \sim 1$, or $D_1 \sim C_1$. In the resonant case $D_1 = 0$ this appears because the possibility of spontaneous emission prevents the cavity field from building up, while in the nonresonant case $D_1 \sim C_1$ it is a consequence of the fact that the driving of the transition from $|1\rangle$ to $|e\rangle$ is inefficient when the atom-light detuning is large. The second factor is also small if $D_1 \gg C_1$, but this is uninteresting, since in that limit the atom does not interact significantly with the field and $F_1^1 \approx F_1^0 \approx 1$. Regarding the third factor, we note that t_3 , r_3 , η_3 , P , F_1^1 , and $F_2^{i_2}$ all have a norm, which is smaller than or equal to one. The factor is of order unity or smaller unless r_3 , η_3 , and $P F_1^1 F_2^{i_2}$ are all close to plus one, which corresponds to the situation of constructive interference in the loop. We thus require $|1 - r_3 P F_1^1 F_2^{i_2} \sqrt{\eta_3}| \gg C_1^{-1}$.

From a semiclassical point of view the problem in having a too large input field to cavity 1 is that the nonlinearity of the Maxwell-Bloch equations for the cavity field can no longer be neglected and this can give rise to complicated dynamics such as quantum jumps at random times between two different quasi steady states [22, 23]. For the same set of parameters, it is, for instance, possible to have a low amplitude of the cavity field and a low excitation of the atom or to have a large cavity field amplitude and a strong driving of the atom between $|1\rangle$ and $|e\rangle$ [24, 25]. The former possibility is similar to the case

of a low input field considered in the present paper, but, except for the spontaneous emission events, the latter is similar to the situation where the atom is in the state $|0\rangle$. It is then difficult to distinguish $|0\rangle$ and $|1\rangle$, and the setup is no longer suitable for a parity measurement. The condition for cavity 2 adds the requirements $C_2 \gg 1$ and $|1 - r_3 P F_1^{i_1} F_2^1 \sqrt{\eta_3}| \gg C_2^{-1}$. We note that cooperativity parameters around hundred have been achieved in experiments, see for instance [26–28], and the above approximation is thus realistic.

III. PURITY DECAY IN THE ODD AND EVEN PARITY SUBSPACES

In the considered parity measurement scheme there is no mechanism, which transforms any of the four possible two-qubit states into each other, but the measurement gradually projects the state onto either the even or the odd subspace. Ideally, the measurement is unable to distinguish states within the odd subspace and to distinguish states within the even subspace, and a main indication of the quality of the measurement is thus the final purity of an initially pure state in either the odd or the even subspace after a measurement time, which is sufficient to distinguish odd parity states from even parity states.

Considering the odd subspace, the state of the system at time t is given by (5) with the additional requirements $i_1 \neq i_2$ and $j_1 \neq j_2$. The entanglement between the atoms and the cavity fields reduces the purity of the atomic state during the measurement, but this purity is regained, when the input field is turned off and the cavity fields decay to the vacuum state. While the rate of non-regainable loss of purity may be slightly different during the initial and final transients, we do not expect anything dramatic to happen, since the transient dynamics does not break the symmetry, which ensures that the odd parity states are indistinguishable in the output field. For a total measurement time t_m , which is long compared to the transients, we may thus concentrate on the rate of loss of purity in steady state. (This is a matter of $|\alpha|^2$ being sufficiently small, since the duration of the transients does not depend on α , while $t_m^{-1} \propto |\alpha|^2$ as we shall find below.)

In the time interval from t to $t + dt$ several interactions take place. The coherent state $|\xi_1^{10} \sqrt{\tau_1}\rangle$ in cavity 1 interacts with the atom and an average number of $2C_1 \kappa_1 dt |\xi_1^{10}|^2 \tau_1 / (1 + D_1^2)$ photons are lost through spontaneous emission. The loss can be modeled as a beam splitter, which transforms $|\xi_1^{10} \sqrt{\tau_1}\rangle |\text{vac}\rangle$ into $|\sqrt{\eta_{\text{SE},1}} \xi_1^{10} \sqrt{\tau_1}\rangle |\sqrt{1 - \eta_{\text{SE},1}} \xi_1^{10} \sqrt{\tau_1}\rangle$, where $1 - \eta_{\text{SE},1} \equiv 2C_1 \kappa_1 dt / (1 + D_1^2)$, followed by a trace over the last mode. In steady state, $|\sqrt{\eta_{\text{SE},1}} \xi_1^{10} \sqrt{\tau_1}\rangle$ is transformed back into $|\xi_1^{10} \sqrt{\tau_1}\rangle$ before the next interaction with the atom. The

overall transformation is thus

$$c_{1001} \rightarrow c_{1001} \langle 0 | \sqrt{1 - \eta_{\text{SE},1}} \xi_1^{10} \sqrt{\tau_1} \rangle, \quad (12)$$

$$c_{0110} \rightarrow c_{0110} \langle \sqrt{1 - \eta_{\text{SE},1}} \xi_1^{10} \sqrt{\tau_1} | 0 \rangle, \quad (13)$$

while c_{1100} and c_{0011} are unchanged. Spontaneous emission in cavity 2 leads to a similar transformation, and loss through the lower mirror in Fig. 1 gives rise to a factor $\langle \sqrt{1 - \eta_3} \zeta_3^{01} \sqrt{dt} | \sqrt{1 - \eta_3} \zeta_3^{10} \sqrt{dt} \rangle$ on c_{1001} and the complex conjugate factor on c_{0110} . Finally, homodyne detection of the p -quadrature of the output field leaves the state unchanged if $\text{Im}(\beta^{10}) = \text{Im}(\beta^{01})$. In fact, if $\beta^{10} = \beta^{01}$, it does not even lead to decoherence if the output field is simply traced out, which is why we do not consider detection inefficiency here.

Collecting the factors for the whole measurement from 0 to t_m and neglecting transients, the final purity of the atomic state after the measurement evaluates to

$$\text{Tr}(\rho_{\text{at}}(t \rightarrow \infty)^2) \approx c_{1100}(0)^2 + c_{0011}(0)^2 + 2|c_{1001}(0)|^2 \times \exp \left[- \left(\nu_{\text{odd},1}^{(\text{SE})} + \nu_{\text{odd},2}^{(\text{SE})} + \nu_{\text{odd}}^{(\text{L})} \right) t_m \right], \quad (14)$$

where $\rho_{\text{at}}(t)$ is the density operator of the state of the atoms obtained by tracing out the light fields, $\nu_{\text{odd},q}^{(\text{SE})} = 2C_q \kappa_q |\xi_q^{\delta_{1,q} \delta_{2,q}}|^2 \tau_q / (1 + D_q^2)$ is the rate of spontaneous emission events for atom q , and $\nu_{\text{odd}}^{(\text{L})} = (1 - \eta_3) |\zeta_3^{10} - \zeta_3^{01}|^2$ represents the rate at which it would be possible to distinguish the two states within the odd subspace if the light lost at the lower mirror in Fig. 1 was detected. Additional photon losses in cavity 1 and 2 could be taken into account by including a factor of $\sqrt{\eta_q}$ in (6) and adding the terms $(1 - \eta_q) |\xi_q^{10} - \xi_q^{01}|^2 t_m$, $q = 1, 2$, to the exponent in (14), where $(1 - \eta_q)$ is the fraction of photons lost in cavity q per round trip. In appendix B, we show that the rate of gain of information in homodyne detection is of order the square of the distance between the conditional amplitudes of the output field projected onto the direction of the measured quadrature, and for $\alpha = \alpha^*$ and $\text{Im}(\beta^{10}) = \text{Im}(\beta^{01})$, the relevant size of t_m is thus $t_m = \max(t_m^{00}, t_m^{11})$, where $t_m^{ii} \equiv |\text{Im}(\beta^{10}) - \text{Im}(\beta^{ii})|^{-2}$.

If $\text{Im}(\beta^{00}) = \text{Im}(\beta^{11})$, the final purity of a state restricted to the even subspace is likewise

$$\text{Tr}(\rho_{\text{at}}(t \rightarrow \infty)^2) \approx c_{1111}(0)^2 + c_{0000}(0)^2 + 2|c_{1010}(0)|^2 \times \exp \left[- \left(\nu_{\text{even},1}^{(\text{SE})} + \nu_{\text{even},2}^{(\text{SE})} + \nu_{\text{even}}^{(\text{L})} \right) t_m \right], \quad (15)$$

where $\nu_{\text{even},q}^{(\text{SE})} = 2C_q \kappa_q |\xi_q^{11}|^2 \tau_q / (1 + D_q^2)$ and $\nu_{\text{even}}^{(\text{L})} = (1 - \eta_3) |\zeta_3^{11} - \zeta_3^{00}|^2$. For $\text{Im}(\beta^{00}) \neq \text{Im}(\beta^{11})$, the two-qubit state is projected onto either $|00\rangle$ or $|11\rangle$ at a rate, which is of order $|\text{Im}(\beta^{00}) - \text{Im}(\beta^{11})|^2 \sim C^{-2}$, and we need to multiply $|c_{1111}(0)|^2$, $|c_{0000}(0)|^2$, and $|c_{1010}(0)|^2$ by the square of the accumulated weight factors arising due to the measurement.

IV. NONRESONANT LIGHT-ATOM COUPLING

In the following, we compare the performance of the setup in Fig. 1 with $r_3 \neq 0$ and an open loop setup with $r_3 = 0$. We first consider a nonresonant light-atom interaction with $D \sim C$ and $C \gg 1$, assuming $D_1 = D_2 = D$ and $C_1 = C_2 = C$. In this case, the primary effect of the light-atom interaction is a phase shift of the light field, and we apply the approximation $F_1^1 = F_2^1 \approx (D + iC)/(D - iC) \equiv F$. Furthermore, $F_1^0 = F_2^0 = F^*$ such that

$$t_m^{-1} = \frac{\eta_3(1 - r_3^2)^2(1 - \sqrt{\eta_3}r_3)^2(1 - \text{Re}(F^2))^2|\alpha|^2}{(1 + \sqrt{\eta_3}r_3)^2(1 + 2\sqrt{\eta_3}r_3\text{Re}(F^2) + \eta_3r_3^2)^2}, \quad (16)$$

$$\nu_{\text{odd},1}^{(\text{SE})} t_m = \frac{8C(1 + 2\sqrt{\eta_3}r_3\text{Re}(F^2) + \eta_3r_3^2)^2}{\eta_3(C^2 + D^2)(1 - r_3^2)(1 - \sqrt{\eta_3}r_3)^2(1 - \text{Re}(F^2))^2}, \quad (17)$$

$$\nu_{\text{even},1}^{(\text{SE})} t_m = \frac{8C(1 + \sqrt{\eta_3}r_3)^2(1 + 2\sqrt{\eta_3}r_3\text{Re}(F^2) + \eta_3r_3^2)}{\eta_3(C^2 + D^2)(1 - r_3^2)(1 - \sqrt{\eta_3}r_3)^2(1 - \text{Re}(F^2))^2}, \quad (18)$$

$$\nu_{\text{odd}}^{(\text{L})} t_m = \frac{4\text{Im}(F)^2(1 - \eta_3)(1 + 2\sqrt{\eta_3}r_3\text{Re}(F^2) + \eta_3r_3^2)^2}{\eta_3(1 - \text{Re}(F^2))^2(1 - r_3^2)(1 - \sqrt{\eta_3}r_3)^2}, \quad (19)$$

and

$$\nu_{\text{even}}^{(\text{L})} t_m = \frac{4\text{Im}(F)^2(1 - \eta_3)(1 + \sqrt{\eta_3}r_3)^2}{\eta_3(1 - \text{Re}(F^2))^2(1 - r_3^2)}. \quad (20)$$

The latter four quantities are plotted in Fig. 2, and we note that $\nu_{\text{odd},2}^{(\text{SE})} = \eta_3 \nu_{\text{odd},1}^{(\text{SE})}$ and $\nu_{\text{even},2}^{(\text{SE})} = \eta_3 \nu_{\text{even},1}^{(\text{SE})}$.

The figure shows that it is possible in the odd subspace to reduce both the rate of spontaneous emission events and the rate of decoherence due to other light field losses relative to the rate of gain of information due to the measurement by choosing a nonzero value of r_3 , but only if D/C is sufficiently close to plus or minus one, which corresponds to the situation of constructive interference in the loop for the even subspace. Note that $\nu_{\text{odd},q}^{(\text{SE})}$ and $\nu_{\text{odd}}^{(\text{L})}$ depend on r_3 in the same way, and for a given value of $D/C \in [-\sqrt{3}, -1/\sqrt{3}] \cup [1/\sqrt{3}, \sqrt{3}]$ and $\eta_3 = 1$, the optimal choice of beam splitter reflectivity is given by

$$r_{3,\text{opt}} = \frac{\sqrt{3 - 3\text{Re}(F^2)^2 - 2 - \text{Re}(F^2)}}{1 + 2\text{Re}(F^2)}. \quad (21)$$

Outside this interval, it is optimal to choose $r_3 = 0$. As far as light field losses in the loop are concerned, the smallest losses occur for $|D|/C = 1$, and Fig. 2(a)

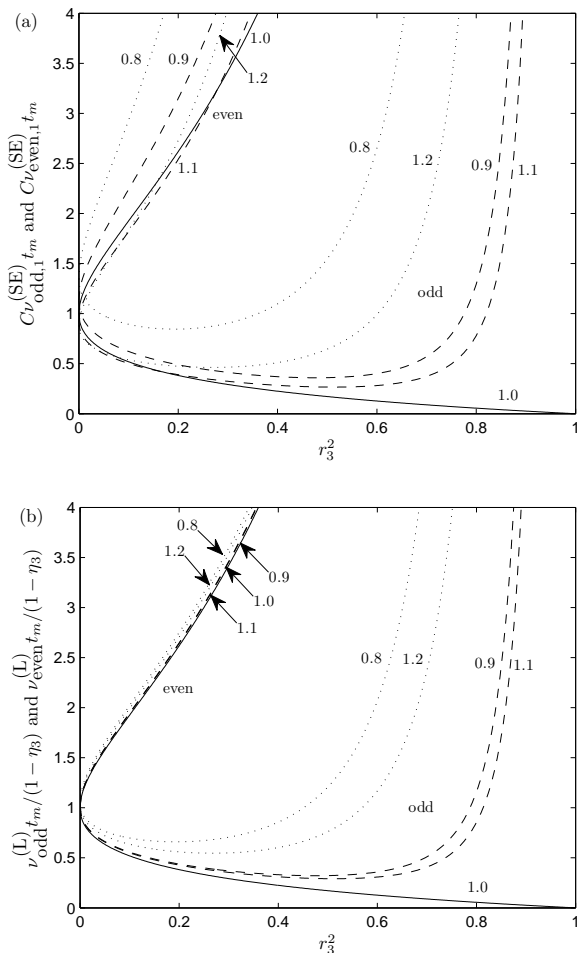


FIG. 2: (a) rate of spontaneous emission events relative to the rate of gain of information due to the measurement in units of C^{-1} and (b) rate of decoherence within the even and odd subspaces due to loss at the lower mirror in Fig. 1 relative to the measurement rate in units of $1 - \eta_3$ for a nonresonant light-atom interaction. Both graphs are for $\eta_3 = 1$ and the different curves correspond to $D/C = 0.8, 0.9, 1.0, 1.1,$ and 1.2 as indicated. The curves in the upper left region are for the even subspace and the lower curves are for the odd subspace.

suggests that this is also a good choice to reduce the total number of spontaneous emission events (the optimal value of $|D|/C$ with respect to loss due to spontaneous emission varies from $\sqrt{2}$ for $\sqrt{\eta_3}r_3 = 0$ to 1 for $\sqrt{\eta_3}r_3 = 1$). For $|D|/C \rightarrow 1$, $r_{3,\text{opt}} \rightarrow 1$, but since $\sqrt{\eta_3}PF_1^1F_2^1 = +1$ for the even subspace, the results are only valid if $1 - r_3 \gg C^{-1}$. In fact, if $\eta_3 < 1$ or C is not infinite, the curves go to infinity for $r_3 \rightarrow 1$. This happens because $t_m \rightarrow \infty$, which in turn is a consequence of the fact that it is only possible to couple light into the system for $r_3 = 1$ if there is perfect constructive interference within the loop and no loss, and the output field amplitude is hence $i\alpha$ for all states of the qubits if this is not the case. In conclusion, a reasonable strategy is

to choose the light-atom detuning such that $|D|/C = 1$, and then compute the optimal value of r_3 from the above expressions if the relative photon loss $1 - \eta_3$ in the loop is large compared to C^{-1} . Otherwise one has to optimize r_3 under the constraint $|1 + r_3F_1^1F_2^1\sqrt{\eta_3}| \gg C^{-1}$. For the even subspace, on the other hand, it is always optimal to choose $r_3 = 0$. Finally, we note that the total number of spontaneous emission events is of order C^{-1} , while the contribution to the exponent in (14) from loss at the lower mirror in Fig. 1 is of order $(1 - \eta_3)$. The relative importance of the two effects is thus given by $C(1 - \eta_3)$. The importance of additional photon losses in cavity 1 or 2 relative to spontaneous emission is likewise given by $C(1 - \eta_q)$.

V. RESONANT LIGHT-ATOM COUPLING

For a resonant light-atom coupling $D = 0$ and identical atom and cavity parameters, $F_1^1 = F_2^1 = (1 - 2C)/(1 + 2C) \equiv G$ and $F_1^0 = F_2^0 = 1$. In this case,

$$t_m^{-1} = (t_m^{11})^{-1} = \frac{\eta_3 G^2 (G - 1)^2 (1 - r_3^2)^2 |\alpha|^2}{(1 - \sqrt{\eta_3} r_3 G)^2 (1 - \sqrt{\eta_3} r_3 G^2)^2}, \quad (22)$$

$$\nu_{\text{odd},1}^{(\text{SE})} t_m = \frac{8C(1 - \sqrt{\eta_3} r_3 G^2)^2}{(1 + 2C)^2 \eta_3 G^2 (G - 1)^2 (1 - r_3^2)}, \quad (23)$$

$$\nu_{\text{even},1}^{(\text{SE})} t_m = \frac{8C(1 - \sqrt{\eta_3} r_3 G)^2}{(1 + 2C)^2 \eta_3 G^2 (G - 1)^2 (1 - r_3^2)}, \quad (24)$$

$$\nu_{\text{odd}}^{(\text{L})} t_m = \frac{(1 - \eta_3)(1 - \sqrt{\eta_3} r_3 G^2)^2}{\eta_3 G^2 (1 - r_3^2)}, \quad (25)$$

and

$$\nu_{\text{even}}^{(\text{L})} t_m = \frac{(1 - \eta_3)(1 - \eta_3 r_3^2 G^2)^2}{\eta_3 G^2 (1 - r_3^2)(1 - \sqrt{\eta_3} r_3)^2}, \quad (26)$$

while $\nu_{\text{odd},2}^{(\text{SE})} = \eta_3 \nu_{\text{odd},1}^{(\text{SE})}$ and $\nu_{\text{even},2}^{(\text{SE})} = \eta_3 G^2 \nu_{\text{even},1}^{(\text{SE})}$. These results are illustrated for $\eta_3 = 1$ in Fig. 3, which shows that the total decoherence in the odd subspace also in this case can be reduced by choosing a nonzero r_3 . For a given value of C and η_3 , the minimum appears at $r_{3,\text{opt}} = \sqrt{\eta_3} G^2$, but again one should keep the constraint $|1 - r_3 F_1^1 F_2^1 \sqrt{\eta_3}| \gg C^{-1}$ in mind.

Assuming $D = C$ and $C \rightarrow \infty$ for the nonresonant case and $C \rightarrow \infty$ for the resonant case, we observe that the expressions for t_m are the same, and the two possibilities are thus equally efficient in distinguishing the even and the odd subspaces. We also note that $\zeta_1^{i_1 i_2}$ is the same for the resonant and the nonresonant coupling, leading to the same values for the rate of decoherence due to light field losses in the loop. The rates of spontaneous emission are also the same except for a factor of 2, which arises from the details of the light-atom interaction: For

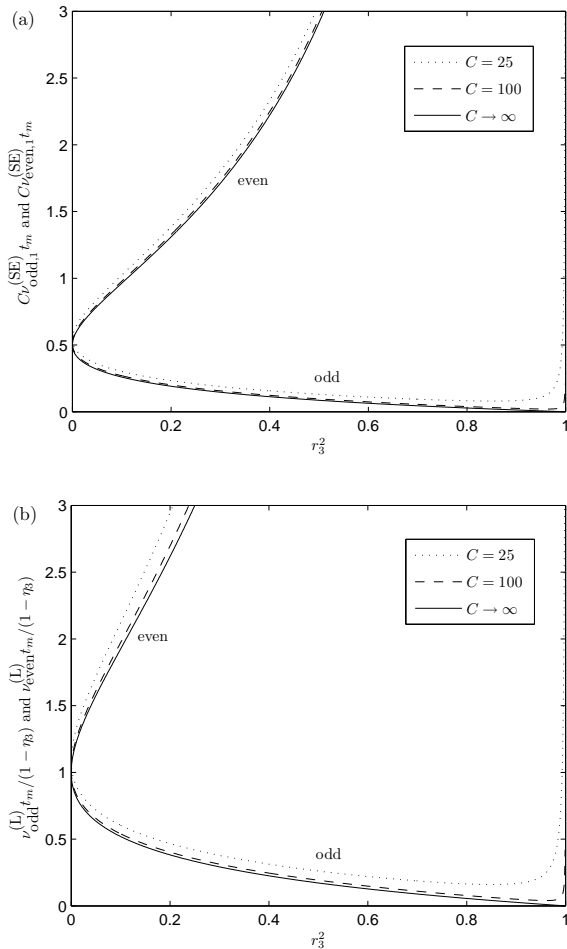


FIG. 3: (a) rate of spontaneous emission events relative to the measurement rate in units of C^{-1} and (b) rate of decoherence within the even and odd subspaces due to loss at the lower mirror in Fig. 1 relative to the measurement rate in units of $1 - \eta_3$ for a resonant light-atom interaction and $\eta_3 = 1$. The different curves are for different values of the cooperativity parameter as shown in the legend. The curves to the upper left are for the even subspace and the lower curves are for the odd subspace.

cavities containing an atom in the state $|1\rangle$, the number of photons in the cavity is a factor of $1/(2C^2)$ smaller in the resonant case compared to the nonresonant case, but the fraction of photons lost per round trip is a factor of C^2 larger. Finally, the significance of additional photon losses in cavity 1 and 2 is the same for resonant and nonresonant coupling because $|\xi_q^{10} - \xi_q^{01}|$ is the same and $|\xi_q^{11} - \xi_q^{00}|$ is the same. The performance is thus roughly the same for a resonant and a nonresonant light-atom interaction.

VI. CONCLUSION

In conclusion, we have analyzed the performance of a parity measurement on two qubits, each encoded in two ground state levels of a single atom in a cavity. The measurement relies on the phase shift imposed conditionally on a continuous beam of light, which interacts sequentially with the two qubits. We have found that the rate of decoherence within the odd parity subspace due to spontaneous emission and light field losses can be decreased by allowing the probe field to interact several times with the two qubits before it is detected, but the decrease happens at the expense of an increased decoherence rate within the even parity subspace. Finally, we have shown that the performance of the parity measurement is about the same for a resonant and for a nonresonant light-atom interaction if the light-atom detuning is chosen as $|\Delta| = g^2/\kappa$, where g is the light-atom coupling strength and κ is the decay rate of light in the cavity.

The proposed measurement can, for instance, be used to prepare two qubits in an entangled odd parity state with higher purity. Improvements can also be obtained in quantum error correction if the qubit state is encoded in the odd subspace and errors are corrected continuously such that the even subspace is practically avoided. In case of the three-qubit bit-flip correction code, for instance, one would use the three-qubit states $|010\rangle$ and $|101\rangle$ as the code space and then perform the proposed parity measurement on qubits 1 and 2 and on qubits 2 and 3. This could be done either by switching between the two measurements or by using two different polarizations of the light field, one of which interacts with the qubit state $|1\rangle$ and the other with the qubit state $|0\rangle$. Polarizing beam splitters could then be used to guide the polarization components into the right cavities.

We have presented the proposed structure as a closed loop setup, but we note the similarities with both a coherent feedback loop and a cavity array. The idea of having cavities within cavities and utilizing constructive and destructive interference may be useful in other settings as well.

APPENDIX A: SYMMETRY PROPERTIES OF THE OUTPUT FIELD

We show that the output field is the same for the two odd parity states even if the input field is time dependent, provided the cavity and atom parameters are the same for the two cavities. At first, we only require $\tau_1 = \tau_2 \equiv \tau$. We denote the light traveling time from cavity 1 to cavity 2 by T and choose the time $t = 0$ such that $\zeta_2^{i_1 i_2}(t < 0) = 0$, while $\zeta_2^{i_1 i_2}(t)$ may have an arbitrary time dependence

for $t \geq 0$. Inspection of Fig. 1 then gives

$$\begin{aligned} \zeta_5^{i_1 i_2}(T + (n+x)\tau) &= \sum_{q=0}^{n-1} \left(it_1^2 r_2 f_1^{i_1} \sqrt{\eta_3} (r_1 f_1^{i_1})^{n-q-1} \right. \\ &\quad + ir_1 t_2^2 f_2^{i_2} \sqrt{\eta_3} (r_2 f_2^{i_2})^{n-q-1} - it_1^2 t_2^2 f_1^{i_1} f_2^{i_2} \sqrt{\eta_3} \\ &\quad \times \sum_{k=0}^{n-q-2} (r_1 f_1^{i_1})^k (r_2 f_2^{i_2})^{n-q-k-2} \left. \right) \zeta_2^{i_1 i_2}((q+x)\tau) \\ &\quad - ir_1 r_2 \sqrt{\eta_3} \zeta_2^{i_1 i_2}((n+x)\tau), \quad (\text{A1}) \end{aligned}$$

where $n \in \mathbb{N}_0$ and $x \in [0, 1[$. The first term in the brackets is the contribution from the field, which enters into cavity 1 at time $(q+x)\tau$ and travels $n-q$ round trips before it leaves cavity 1 and travels to the input mirror of cavity 2 where it is reflected. The second term is the contribution from the field, which travels zero round trips in cavity 1 and $n-q$ round trips in cavity 2, and the third term is the contribution from the field, which travels $k+1$ round trips in cavity 1 and $n-q-k-1$ round trips in cavity 2. The final term is the contribution from the field, which travels zero round trips in both cavity 1 and 2.

For $r_1 = r_2$ and $t_1 = t_2$, we note that $\zeta_5^{i_1 i_2}$ is invariant under exchange of i_1 and i_2 if $\zeta_2^{i_1 i_2}$ is invariant at all times and $f_1^0 = f_2^0$ and $f_1^1 = f_2^1$. The former requirement is fulfilled if $\zeta_5^{i_1 i_2}$ is invariant, since $\zeta_2^{i_1 i_2}$ is a linear combination of α and $\zeta_5^{i_1 i_2}$, and the latter requirement is fulfilled if the cavity and atom parameters are the same for the two cavities. The invariance of $\beta^{i_1 i_2}$ then follows from the fact that $\beta^{i_1 i_2}$ is a linear combination of α and $\zeta_5^{i_1 i_2}$.

APPENDIX B: RATE OF GAIN OF INFORMATION DUE TO HOMODYNE DETECTION

We estimate the ability of homodyne detection to distinguish continuous coherent states. Consider the light field mode, which is detected in the time interval from

t and $t + dt$. We define the integrated photo current as $dy \equiv k/|\alpha|$, where k is the difference in the observed number of photons at the two detectors within the time interval and $|\alpha|^2$ is the average number of photons per unit time in the local oscillator beam. Following the derivation in [29], and assuming that the measured light field mode is in the coherent state $|\beta\sqrt{dt}\rangle$, we find

$$dy = dW - ie^{-i\theta} \beta dt + ie^{i\theta} \beta^* dt, \quad (\text{B1})$$

where dW is a Gaussian stochastic variable with mean zero and variance dt and θ is the phase of the local oscillator. The mean value of the photo current is thus $\langle\langle dy \rangle\rangle = -ie^{-i\theta} \beta dt + ie^{i\theta} \beta^* dt$, and the standard deviation is \sqrt{dt} .

The measurement time t_m required to distinguish two coherent states with amplitudes β_1 and β_2 may now be defined as the time for which the absolute value of the difference in the mean value of the photo current integrated from 0 to t_m

$$\left| \int_0^{t_m} \langle\langle dy_1 \rangle\rangle - \int_0^{t_m} \langle\langle dy_2 \rangle\rangle \right| = |-ie^{-i\theta} \beta_1 t_m + ie^{i\theta} \beta_1^* t_m + ie^{-i\theta} \beta_2 t_m - ie^{i\theta} \beta_2^* t_m| \quad (\text{B2})$$

equals twice the standard deviation $2\sqrt{t_m}$, i.e.,

$$t_m = 4 |-ie^{-i\theta} \beta_1 + ie^{i\theta} \beta_1^* + ie^{-i\theta} \beta_2 - ie^{i\theta} \beta_2^*|^{-2}. \quad (\text{B3})$$

The rate of gain of information is thus the square of the distance in phase space between the two coherent state amplitudes projected onto the direction of the measured quadrature.

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