Adiabatic elimination in quantum stochastic models

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Abstract

We consider a physical system with a coupling to bosonic reservoirs via a quantum stochastic differential equation. We study the limit of this model as the coupling strength tends to infinity. We show that in this limit the solution to the quantum stochastic differential equation converges strongly to the solution of a limit quantum stochastic differential equation. In the limiting dynamics the excited states are removed and the ground states couple directly to the reservoirs.

1 Introduction

It is a frequent occurence in physics to have a system that spends a very limited amount of time in its excited states. This is, for instance, the case if the system is strongly coupled to a low temperature environment (e.g. the optical field). The strong coupling ensures that excitations above the ground levels of the system quickly dissipate into its environment. It is therefore reasonable to ask for a model in which the excited states are eliminated from the description. That is, we would like to have a description that only involves the ground states of a system and its environment. The procedure for going from the full model to the reduced model is called *adiabatic elimination*.

We study adiabatic elimination in the context of quantum stochastic models [15] which arise by taking a weak coupling limit of QED (quantum electrodynamics) models [1, 13, 5], and are widely applicable to systems studied in quantum optics. Specifically, quantum stochastic models are the starting point for deriving master equations, filtering equations, and input-output relations. In the quantum optics community adiabatic elimination is a common technique, used, for instance, in atomic systems [22, 2, 6, 11] and in cavity QED models [12, 23] as well as in more recent work on quantum feedback [7, 9, 24]. Rigorous results have been demonstrated for adiabatic elimination outside of the quantum stochastic models we consider [19, 4, 11]. At present, however, apart from the work [14] on the elimination of a leaky cavity (using a Dyson series expansion to prove weak convergence), no rigorous results have been obtained on adiabatic elimination in the context of the quantum stochastic models introduced by Hudson and Parthasarathy [15].

We start by considering a family, indexed by a parameter k, of quantum stochastic differential equations (QSDE's). The parameter k can be interpreted as the coupling strength between the system and its environment. The environment is modelled by a collection of bosonic heat baths in the vacuum representation. We assume that the coefficients of the QSDE are all bounded and satisfy the usual conditions guaranteeing a unique unitary solution [15]. We state further assumptions on the coefficients and show that under these assumptions the solution of the QSDE converges strongly to the solution of a limiting QSDE as k tends to infinity (Theorem 2.1). The limiting QSDE represents the adiabatically eliminated time evolution of the system.

The heart of the proof is a technique introduced by T.G. Kurtz [17] that enables the application of the Trotter-Kato Theorem [21]. This allows us to prove strong convergence of the unitaries using convergence of generators of semigroups rather than convergence of a Dyson series expansion. Convergence is first shown on the vacuum vector of the bosonic reservoirs. We then extend this result to any possible vector in the Hilbert space of the reservoirs by sandwiching the unitaries with Weyl operators and using a density argument.

The remainder of this article is organized as follows. In Section 2 we introduce the system coupled to n bosonic reservoirs in the vacuum representation. We state assumptions on the coefficients of the QSDE and present the main convergence theorem. In Section 3 we discuss four applications of the theorem in the context of examples from atomic physics and cavity QED. Section 4 presents the proof of the main convergence theorem. In Section 5 we discuss our results.

2 The main result

Let $\mathcal{H}$ be a Hilbert space and let n be an element of $\mathbb{N}$. Let $\mathcal{F}$ be the *symmetric Fock space* over $\mathbb{C}^n \otimes L^2(\mathbb{R}^+) \cong L^2(\mathbb{R}^+;\mathbb{C}^n)$, i.e.

$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{m=1}^{\infty} L^2(\mathbb{R}^+;\mathbb{C}^n)^{\otimes_s m}.$$

Physically, the Hilbert space $\mathcal{H}\otimes\mathcal{F}$ describes a system $\mathcal{H}$ coupled to n bosonic reservoirs (e.g. n decay channels in the quantized electromagnetic field). For $f \in L^2(\mathbb{R}^+;\mathbb{C}^n)$, we define the *exponential vector* $e(f)$ in $\mathcal{F}$ by

$$e(f) = 1 \oplus \bigoplus_{m=1}^{\infty} \frac{f^{\otimes m}}{\sqrt{m!}}.$$

Moreover, we define the *coherent vector* $\pi(f)$ to be the exponential vector $e(f)$ normalized to unity, i.e. $\pi(f) = \exp(-\frac{1}{2}\|f\|^2)e(f)$. The *vacuum vector* is defined to be the exponential vector $\Phi = e(0) = 1 \oplus 0 \oplus 0 \ldots$ The expectation with respect to the vacuum vector is denoted by ϕ, i.e. ϕ is a map from $\mathcal{B}(\mathcal{F})$ (the bounded operators on $\mathcal{F}$) to $\mathbb{C}$, given by $\phi(W) = \langle \Phi, W\Phi\rangle$ for all $W \in \mathcal{B}(\mathcal{F})$.

The interaction between the system and the bosonic reservoirs is modelled by a quantum stochastic differential equation (QSDE) in the sense of Hudson and Parthasarathy [15] of the form

$$dU_t^{(k)} = \left\{ \left(S_{ij}^{(k)} - \delta_{ij}\right) d\Lambda_t^{ij} + L_i^{(k)} dA_t^{i\dagger} - L_i^{(k)\dagger} S_{ij}^{(k)} dA_t^j \ + K^{(k)} dt \right\} U_t^{(k)}, \qquad (1)$$

where $U_0^{(k)} = I$. We consistently use the convention that repeated indices that are not within parentheses are being summed (i and j run through $\{1, \dots, n\}$). The Hilbert space adjoint is denoted by a dagger †. We have indexed the equation with a positive number k, and in the following we will be interested in the behaviour of $U_t^{(k)}$ as k tends to infinity. We assume that the following conditions on the coefficients of the QSDE are satisfied.

Assumption 1: For each $k \geq 0$, the coefficients $K^{(k)}, S_{ij}^{(k)}$ and $L_i^{(k)}$ $(i, j \in \{1, \dots, n\})$ of the quantum stochastic differential equation (1) are bounded operators on $\mathcal{H}$. Furthermore, for each $k \geq 0$, the coefficients satisfy the following relations

$$K^{(k)} + K^{(k)\dagger} = -L_i^{(k)\dagger} L_i^{(k)}, \qquad S_{il}^{(k)} S_{jl}^{(k)\dagger} = \delta_{ij} I, \qquad S_{li}^{(k)\dagger} S_{lj}^{(k)} = \delta_{ij} I.$$

Hudson and Parthasarathy [15] show that under Assumption 1, the quantum stochastic differential equation (1) has a unique unitary solution $U_t^{(k)}$, and, the adjoint $U_t^{(k)\dagger}$ satisfies the adjoint of Eq. (1).

Assumption 2: There exist bounded operators Y, A, B, F_i, G_i and W_{ij} (independent of k) on $\mathcal{H}$ such that

$$K^{(k)} = k^2 Y + kA + B, \qquad L_i^{(k)} = kF_i + G_i, \qquad S_{ij}^{(k)} = W_{ij},$$

for all $i, j \in \{1, \dots, n\}$.

We define P_0 as the orthogonal projection onto $\mathrm{Ker}(Y)$. Let $P_1 = I - P_0$ be its complement in $\mathcal{H}$. We use the following notation $\mathcal{H}_0 = P_0\mathcal{H}$ and $\mathcal{H}_1 = P_1\mathcal{H}$. Physically, one should think of $\mathcal{H}_0$ as the ground states and of $\mathcal{H}_1$ as the excited states of the system.

Assumption 3: There exists a bounded operator Y_1^{-1} on $\mathcal{H}$ such that $P_1 Y_1^{-1} = Y_1^{-1} P_1$ and

$$YY_1^{-1} P_1 Z P_0 = P_1 Z P_0, \qquad P_0 X P_1 Y_1^{-1} Y = P_0 X P_1, \qquad (2)$$

where $Z = A, F_i^\dagger W_{ij}$, $(j \in \{1, \dots, n\})$ and $X = A, B, F_i, G_i, W_{ij}, G_i^\dagger W_{ij}, F_i Y_1^{-1} F_j$, $F_i Y_1^{-1} A, F_i Y_1^{-1} F_l^\dagger W_{lj}, AY_1^{-1} A, AY_1^{-1} F_i, AY_1^{-1} F_l^\dagger W_{lj}$, $(i, j \in \{1, \dots, n\})$. Moreover, for all $i, j \in \{1, \dots, n\}$ the following products are zero

$$P_0 Y P_1 = P_0 A P_0 = F_i P_0 = P_0(\delta_{il} + F_i Y_1^{-1} F_l^\dagger) W_{lj} P_1 = 0.$$

Note that the existence of Y_1^{-1} satisfying the assumptions in Eq. (2) is immediate if Y maps $\mathcal{H}_1$ surjectively onto $\mathcal{H}_1$ and is therefore invertible on $\mathcal{H}_1$.

Definition 1: Suppose Assumption 2 and 3 hold. We define for all $i, j \in \{1, \ldots, n\}$ the following bounded operators on $\mathcal{H}$

$$
\begin{aligned}
&K = P_0 \left(B - AY_1^{-1}A\right) P_0, \qquad L_i = \left(G_i - F_i Y_1^{-1} A\right) P_0, \\
&S_{ij} = \left(\delta_{il} + F_i Y_1^{-1} F_l^\dagger\right) W_{lj} P_0.
\end{aligned}
$$

Assumption 4: For all $i, j \in \{1, \ldots, n\}$ the following products are zero

$$
P_1 L_i = P_1 S_{ij} = 0.
$$

Lemma 1: *Suppose that Assumption 1, 2, 3 and 4 hold. The operators in Definition 1 satisfy*

$$
K + K^\dagger = -L_i^\dagger L_i, \qquad S_{il} S_{jl}^\dagger = \delta_{ij} P_0, \qquad S_{li}^\dagger S_{lj} = \delta_{ij} P_0.
$$

Proof. By Assumptions 1 and 2 we have $K^{(k)} + K^{(k)} = -L_i^{(k)\dagger} L_i^{(k)}$, $K^{(k)} = k^2 Y + kA + B$ and $L_i^{(k)} = kF_i + G_i$ for all $k \geq 0$. Moreover, $F_i P_0 = 0$, by Assumption 3. Combining these results leads to

$$
\begin{aligned}
&- F_i^\dagger F_i = Y + Y^\dagger \\
&- P_1 F_i^\dagger G_i P_0 = P_1 (A + A^\dagger) P_0 \\
&- P_0 G_i^\dagger G_i P_0 = P_0 (B + B^\dagger) P_0.
\end{aligned} \tag{3}
$$

We then use $Y Y_1^{-1} A P_0 = A P_0$ from Assumption 3 and L_i from Definition 1 to derive

$$
\begin{aligned}
-L_i^\dagger L_i &= -P_0 (G_i^\dagger - A^\dagger Y_1^{-1\dagger} F_i^\dagger)(G_i - F_i Y_1^{-1} A) P_0 \\
&= P_0 (B + B^\dagger) P_0 - P_0 A^\dagger Y_1^{-1\dagger} (A + A^\dagger) P_0 \\
&\quad - P_0 (A + A^\dagger) Y_1^{-1} A P_0 + P_0 A^\dagger (Y_1^{-1\dagger} + Y_1^{-1}) A P_0 \\
&= P_0 (B + B^\dagger) P_0 - P_0 A Y_1^{-1} A P_0 \; - P_0 A^\dagger Y_1^{-1\dagger} A^\dagger P_0 \\
&= P_0 (K + K^\dagger) P_0.
\end{aligned}
$$

By Definition 1

$$
S_{ij} = \left(\delta_{il} + F_i Y_1^{-1} F_l^\dagger\right) W_{lj} P_0.
$$

Combining this with $-F_i^\dagger F_i = Y + Y^\dagger$ from above,

$$
\begin{aligned}
S_{li}^\dagger S_{lj} &= P_0 W_{mi}^\dagger \left(\delta_{ml} + F_m Y_1^{-1\dagger} F_l^\dagger\right) \left(\delta_{ln} + F_l Y_1^{-1} F_n^\dagger\right) W_{nj} P_0 \\
&= P_0 W_{li}^\dagger W_{lj} P_0 = P_0 \delta_{ij}.
\end{aligned}
$$

Then use $P_0\left(\delta_{il}+F_iY_1^{-1}F_l^\dagger\right)W_{lj}P_1 = 0$ from Assumption 3 and $P_1S_{ij}P_0 = 0$ from Asumption 4 to derive

$$\begin{aligned}S_{il}S_{jl}^\dagger &= P_0\left(\delta_{in}+F_iY_1^{-1}F_n^\dagger\right)W_{nl}W_{ml}^\dagger\left(\delta_{mj}+F_mY_1^{-1\dagger}F_j^\dagger\right)P_0\\ &= P_0\left(\delta_{in}+F_iY_1^{-1}F_n^\dagger\right)\left(\delta_{nj}+F_nY_1^{-1\dagger}F_j^\dagger\right)P_0=\delta_{ij}P_0.\end{aligned}$$

□

The operators given by Definition 1 are the coefficients of a QSDE on the Hilbert space $\mathcal{H}\otimes\mathcal{F}$

$$dU_t=\left\{\left(S_{ij}-\delta_{ij}P_0\right)d\Lambda_t^{ij}+L_idA_t^{i\dagger}-L_i^\dagger S_{ij}dA_t^j\ +Kdt\right\}U_t,\qquad U_0=I. \tag{4}$$

Lemma 1 implies that under Assumptions 1, 2, 3 and 4, Eq. (4) has a unique unitary solution on $\mathcal{H}$ [15], and, the adjoint $U_t^\dagger$ satisfies the adjoint of Eq. (4). Moreover, U_t maps $\mathcal{H}_0$ to $\mathcal{H}_0$. Note that $U_tP_1 = P_1$.

Theorem 2.1: *Suppose Assumption 1, 2, 3 and 4 hold. Let $U_t^{(k)}$ be the unique unitary solution to Eq. (1). Let U_t be the unique unitary solution to Eq. (4) where the coefficients are given by Definition 1. Then $U_t^{(k)}P_0$ converges strongly to U_tP_0, i.e.*

$$\lim_{k\to\infty}U_t^{(k)}\psi=U_t\psi,\qquad\forall\psi\in\mathcal{H}_0\otimes\mathcal{F}.$$

We prove Theorem 2.1 in Section 4.

3 Examples

We use the following definitions in the first two examples below. Let $(|e\rangle,|g\rangle)$ be an orthogonal basis of $\mathbb{C}^2$. Define the raising and lowering operators in this basis as

$$\sigma_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad\sigma_-=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Define the Pauli operators

$$\sigma_x=\sigma_++\sigma_-,\qquad\sigma_y=-i\sigma_++i\sigma_-,\qquad\sigma_z=\sigma_+\sigma_--\sigma_-\sigma_+,$$

and define the projectors

$$P_e=\sigma_+\sigma_-,\qquad P_g=\sigma_-\sigma_+.$$

Example 1: (A two-level atom driven by a laser) The Hilbert space for a two-level atom is $\mathcal{H} = \mathbb{C}^2$, with $|e\rangle$ the excited state, and $|g\rangle$ the ground state. Define the detuning $\Delta \in \mathbb{R}$, the decay rate $\gamma \geq 0$ and the complex amplitude $\alpha \in \mathbb{C}$. The QSDE for this system in the electric dipole and rotating wave approximations is [2]

$$dU_t^{(k)} = \Big\{ k\sqrt{\gamma}\sigma_- dA_t^\dagger - k\sqrt{\gamma}\sigma_+ dA_t - ik\alpha\sigma_+ dt - ik\bar{\alpha}\sigma_- dt \\ - \frac{k^2\gamma}{2}\sigma_+\sigma_- dt - ik^2\Delta\sigma_+\sigma_- dt \Big\} U_t^{(k)}, \qquad U_0^{(k)} = I.$$

Define the operators Y, A, B, F, G, W as

$$Y = (-i\Delta - \gamma/2)\sigma_+\sigma_-, \qquad A = -i\alpha\sigma_+ - i\alpha\sigma_-, \qquad B = 0,$$
$$F = \sqrt{\gamma}\sigma_-, \qquad G = 0, \qquad W = I.$$

This satisfies Assumptions 1 and 2, and $P_0 = P_g$. We take $Y_1^{-1} = -(i\Delta + \gamma/2)^{-1}\sigma_+\sigma_-$, and Assumption 3 holds by inspection. Definition 1 leads to the following coefficients

$$K = -\frac{|\alpha|^2}{i\Delta + \gamma/2}P_g, \qquad L = -i\frac{\alpha\sqrt{\gamma}}{i\Delta + \gamma/2}P_g, \qquad S = \frac{i\Delta - \gamma/2}{i\Delta + \gamma/2}P_g.$$

Note that $P_1 L = P_1 S = 0$ satisfying Assumption 4. Theorem 2.1 then shows that $U_t^{(k)} P_0$ converges strongly to $U_t P_0$, given by

$$dU_t = \frac{P_g}{i\Delta + \gamma/2}\Big\{ -\gamma d\Lambda_t - i\alpha\sqrt{\gamma}dA_t^\dagger + i\bar{\alpha}\sqrt{\gamma}dA_t - |\alpha|^2 dt \Big\} U_t, \qquad U_0 = I.$$

In the case that $\gamma = 0$ the two level atom decouples from the field. In this case we may explicitly calculate the ground state evolution as

$$P_0 e^{-i\left(k\alpha\sigma_+ + k\bar{\alpha}\sigma_- + k^2\Delta\sigma_+\sigma_-\right)t} P_0 = \frac{e^{-ik^2\Delta t/2}}{\Omega}\left(\Omega\cos(k\Omega t) + ik\Delta\sin(k\Omega t)\right),$$

with $\Omega = \sqrt{\Delta^2 k^2 + 4|\alpha|^2}$. For $k \to \infty$ this expression limits to $e^{i|\alpha|^2/\Delta}$ which is the solution to our eliminated differential equation $dU_t = i\frac{|\alpha|^2}{\Delta}U_t dt,\ U_0 = I$.

Example 2: (Alkali atom) Now consider a system with Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. Physically, the system represents an alkali atom with no nuclear spin coupled to a driving field on the $S_{1/2} \to P_{1/2}$ transition. We have four orthogonal states in this system corresponding to the atomic excited and ground states with angular momentum $m_z = \pm\frac{1}{2}$ along the z-axis. We define a detuning $\Delta \in \mathbb{R}$, a decay rate $\gamma \geq 0$ and a magnetic field $B_i \in \mathbb{R},\ i \in x, y, z$. The system may emit into $n = 3$ independent dipole modes, A_t^i, where the modes are labelled by $i \in \{x, y, z\}$. The QSDE for this system in the dipole and rotating wave approximations is [2],

$$dU_t^{(k)} = \Big\{ k\sqrt{\gamma}\sigma_- \otimes \sigma_i dA_t^{i\dagger} - k\sqrt{\gamma}\sigma_+ \otimes \sigma_i dA_t^i - \frac{3k^2\gamma}{2}P_e \otimes I dt \\ - i\left(k^2\Delta P_e \otimes I + I \otimes B_i\sigma_i\right) dt \Big\} U_t^{(k)}, \qquad U_0^{(k)} = I.$$

Defining the operators $Y, A, B, F_i, G_i, W_{ij}$ as

$$Y = \left(-i\Delta - \frac{3\gamma}{2}\right) P_e \otimes I, \qquad A = 0, \qquad B = -iI \otimes B_i\sigma_i$$
$$F_i = \sqrt{\gamma}\sigma_- \otimes \sigma_i, \qquad G_i = 0, \qquad W_{ij} = \delta_{ij},$$

satisfies Assumptions 1 and 2, and $P_0 = P_g \otimes I$. We take $Y_1^{-1} = -(i\Delta + \frac{3\gamma}{2})^{-1} P_e \otimes I$, and Assumption 3 holds by inspection. Define the eliminated coefficients as

$$K = -iP_g \otimes B_i\sigma_i, \qquad L_i = 0, \qquad S_{ij} = P_g \otimes \left(\delta_{ij} I - \frac{\gamma}{i\Delta + \frac{3\gamma}{2}} \sigma_i\sigma_j\right).$$

This satisfies Assumption 4. Theorem 2.1 then shows that $U_t^{(k)} P_0$ converges strongly to $U_t P_0$, given by

$$dU_t = P_g \otimes \left\{-iB_i\sigma_i dt - \frac{\gamma}{i\Delta + \frac{3\gamma}{2}} \sigma_i\sigma_j d\Lambda_t^{ij}\right\} U_t, \qquad U_0 = I.$$

In the following two examples we make use of a truncated harmonic oscillator. We have truncated the oscillator to satisfy the boundedness condition of Assumption 1 in the following two examples. Let N be an element in $\mathbb{N}$ such that $N \geq 2$. The Hilbert space of the oscillator is $\mathbb{C}^N$. We choose an orthonormal basis $(|0\rangle, \ldots, |N-1\rangle)$ in $\mathbb{C}^N$. The *annihilation operator* $b: \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$b|n\rangle = \sqrt{n}|n-1\rangle, \qquad n \in \{1, \ldots, N-1\},$$

and $b|0\rangle = 0$. The *creation operator* is defined to be the adjoint $b^\dagger$.

Example 3: (Gough and Van Handel [14]) Let $\mathfrak{h}$ be a Hilbert space. We define $\mathcal{H} = \mathfrak{h} \otimes \mathbb{C}^N$. The Hilbert space $\mathfrak{h}$ describes a system inside a cavity. We model the cavity as a truncated oscillator $\mathbb{C}^N$. Let $E_{ij},\ i, j \in \{0, 1\}$ be bounded operators on $\mathfrak{h}$ such that $E_{ij}^\dagger = E_{ji}$ and $\|E_{11}\| < \frac{\gamma}{2}$. Consider the following QSDE

$$dU_t^{(k)} = \left\{\sqrt{\gamma} k b dA_t^\dagger - \sqrt{\gamma} k b^\dagger dA_t - \frac{\gamma k^2}{2} b^\dagger b dt - iH^{(k)} dt\right\} U_t^{(k)}, \qquad U_0^{(k)} = I. \tag{5}$$

Here γ is a real parameter and $H^{(k)}$ is given by

$$H^{(k)} = k^2 E_{11} b^\dagger b + k E_{10} b^\dagger + k E_{01} b + E_{00}.$$

Define operators Y, A, B, F, G, W as

$$Y = \left(-iE_{11} - \frac{\gamma}{2}\right) b^\dagger b, \qquad A = -i\left(E_{10} b^\dagger + E_{01} b\right), \qquad B = -iE_{00},$$
$$F = \sqrt{\gamma} b, \qquad G = 0, \qquad W = I.$$

This satisfies Assumptions 1 and 2 and $P_0 = I_{\mathfrak{h}} \otimes |0\rangle\langle 0|$. Since $\|E_{11}\| < \frac{\gamma}{2}$, the inverse $\left(iE_{11} + \frac{\gamma}{2}\right)^{-1}$ exists. Let $N_1^{-1}: \mathcal{H}_1 \to \mathcal{H}_1$ be the inverse of the restriction of $b^\dagger b$ to $\mathcal{H}_1$.

Taking $Y_1^{-1} = -\left(iE_{11} + \frac{\gamma}{2}\right)^{-1} N_1^{-1} P_1$ satisfies Assumption 3. Definition 1 leads to the following coefficients

$$
\begin{aligned}
K &= -iE_{00}P_0 - E_{01}\frac{1}{iE_{11} + \frac{\gamma}{2}}E_{10}P_0, \\
L &= \frac{-i\sqrt{\gamma}}{iE_{11} + \frac{\gamma}{2}}E_{10}P_0, \qquad S = \frac{iE_{11} - \frac{\gamma}{2}}{iE_{11} + \frac{\gamma}{2}}P_0.
\end{aligned} \tag{6}
$$

These coefficients satisfy Assumption 4. Theorem 2.1 then shows that $U_t^{(k)}P_0$ converges strongly to U_tP_0, where U_t is given by

$$
dU_t = \left\{(S - P_0)d\Lambda_t + LdA_t^\dagger - L^\dagger SdA_t + Kdt\right\}U_t, \qquad U_0 = I.
$$

Remark 1: Note that we consider a truncated oscillator, where [14] treats the full oscillator, and that we prove our result strongly, whereas [14] proves a weak limit. The convergence of the Heisenberg dynamics follows immediately from our strong result. Apart from these points, Example 3 reproduces the result in [14]. Care must be taken when directly comparing the limit equations, since the results in [14] are presented in the interaction picture with respect to the cavity. Under our assumptions, we define $V_t^{(k)}$ as the solution to

$$
dV_t^{(k)} = \left\{\sqrt{\gamma}kbdA_t^\dagger - \sqrt{\gamma}kb^\dagger dA_t - \frac{\gamma k^2}{2}b^\dagger bdt\right\}V_t^{(k)}, \qquad V_0^{(k)} = I.
$$

The unitary in the interaction picture is then given by $\tilde{U}_t^{(k)} = V_t^{(k)\dagger}U_t^{(k)}$, where $U_t^{(k)}$ is given by Eq. (5). Note that due to Theorem 2.1, $V_t^{(k)}P_0$ converges strongly to V_tP_0, where V_t is given by

$$
dV_t = -2P_0d\Lambda_tV_t, \qquad V_0 = I.
$$

This accounts for the sign difference between the coefficients in the equation for $\tilde{U}_t$ presented in [14], and the coefficients in the equation for U_t given by Eq. (6).

Example 4: (Duan and Kimble [8]) We again consider a system inside a cavity, described by the Hilbert space $\mathcal{H} = \mathfrak{h}\otimes\mathbb{C}^N$. The system inside the cavity is a three level atom, i.e. $\mathfrak{h} = \mathbb{C}^3$. Let $(|e\rangle, |+\rangle, |-\rangle)$ be an orthogonal basis in $\mathfrak{h}$. In this basis we define

$$
\sigma_+^{(+)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \sigma_+^{(-)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Moreover define $\sigma_-^{(\pm)} = \sigma_+^{(\pm)\dagger}$ and $P_\pm = \sigma_-^{(\pm)}\sigma_+^{(\pm)}$. The QSDE for a lambda system with one leg $(+ \leftrightarrow e)$ resonantly coupled to the cavity, under the rotating wave approximation in the rotating frame, is,

$$
\begin{aligned}
dU_t^{(k)} = \Bigg\{ &\sqrt{\gamma}kbdA_t^\dagger - \sqrt{\gamma}kb^\dagger dA_t - \frac{\gamma k^2}{2}b^\dagger bdt + \\
&gk^2\left(\sigma_+^{(+)}b - \sigma_-^{(+)}b^\dagger\right)dt + k\left(\sigma_+^{(-)}\alpha - \sigma_-^{(-)}\bar{\alpha}\right)dt\Bigg\}U_t^{(k)}, \qquad U_0^{(k)} = I.
\end{aligned}
$$

Here γ is a positive real parameter and α is a complex parameter. Note that we extend the model from [8] to allow driving on the uncoupled leg ($- \leftrightarrow e$) of the transition. Define operators Y, A, B, F, G, W as

$$Y = -\frac{\gamma}{2} b^\dagger b + g\Big(\sigma_+^{(+)} b - \sigma_-^{(+)} b^\dagger\Big), \qquad A = \Big(\sigma_+^{(-)}\alpha - \sigma_-^{(-)}\bar{\alpha}\Big), \qquad B = 0,$$
$$F = \sqrt{\gamma} b, \qquad G = 0, \qquad W = I.$$

This satisfies Assumptions 1 and 2 and $P_0 = \Big(|+\rangle\langle+| \; + \; |-\rangle\langle-|\Big) \otimes |0\rangle\langle 0|$. We define the following subspaces of $\mathcal{H}$

$$H_n = \mathrm{span}\Big\{|+\rangle \otimes |n\rangle, \, |-\rangle \otimes |n\rangle, \, |e\rangle \otimes |n-1\rangle\Big\}, \qquad n \in \{1, \ldots, N-1\},$$
$$H_N = \mathrm{span}\Big\{|e\rangle \otimes |N-1\rangle\Big\}.$$

Note that $\mathcal{H}_1 = \bigoplus_{n=1}^N H_n$ and that the subspaces H_n ($n \in \{1, \ldots, N\}$) are all invariant under the action of Y. On the subspaces H_n, $n \in \{1, \ldots, N-1\}$, Y is given by

$$Y|_{H_n} = \begin{pmatrix} -\frac{\gamma n}{2} & 0 & -g\sqrt{n} \\ 0 & -\frac{\gamma n}{2} & 0 \\ g\sqrt{n} & 0 & -\frac{\gamma(n-1)}{2} \end{pmatrix},$$

with respect to the basis $(|+\rangle \otimes |n\rangle, |-\rangle \otimes |n\rangle, |e\rangle \otimes |n-1\rangle)$. Moreover, $Y|_{H_N} = -\frac{\gamma(N-1)}{2}$. The inverse is readily computed to be

$$Y|_{H_n}^{-1} = -\frac{1}{d}\begin{pmatrix} \frac{\gamma(n-1)}{2} & 0 & -g\sqrt{n} \\ 0 & \frac{2d}{\gamma n} & 0 \\ g\sqrt{n} & 0 & \frac{\gamma n}{2} \end{pmatrix}, \qquad n \in \{1, \ldots, N-1\},$$

where $d = \frac{\gamma^2 n(n-1)}{4} + g^2 n$. Moreover, $Y|_{H_N}^{-1} = -\frac{2}{\gamma(N-1)}$. We now define $Y_1^{-1} = \oplus_{n=1}^N Y|_{H_n}^{-1} P_1$. This satisfies Assumption 3. Definition 1 leads to the following coefficients

$$K = -\frac{|\alpha|^2\gamma}{2g^2} P_- \otimes |0\rangle\langle 0|, \qquad L = -\frac{\gamma\alpha}{g}\sigma_-^{(+)}\sigma_+^{(-)} \otimes |0\rangle\langle 0|, \qquad S = P_0 - 2P_- \otimes |0\rangle\langle 0|.$$

These operators satisfy Assumption 4. Theorem 2.1 then shows that $U_t^{(k)} P_0$ converges strongly to $U_t P_0$, where U_t is given by

$$dU_t = \Big\{(S - P_0)d\Lambda_t + L dA_t^\dagger - L^\dagger S dA_t + K dt\Big\} U_t, \qquad U_0 = I.$$

Note that the ground state system is a two-level system on which S acts as σ_z.

4 Proof of Theorem 2.1

Definition 2: Suppose Assumptions 1, 2, 3 and 4 hold. Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H}_0)$ be the Banach spaces of all bounded operators on $\mathcal{H}$ and $\mathcal{H}_0$, respectively. We define for all

$t \geq 0$ and $k \geq 0$

$$T_t^{(k)}(X) = \mathrm{id} \otimes \phi \left(U_t^\dagger X U_t^{(k)} \right), \qquad X \in \mathcal{B}(\mathcal{H}),$$
$$T_t(X) = \mathrm{id} \otimes \phi \left(U_t^\dagger X U_t \right), \qquad X \in \mathcal{B}(\mathcal{H}_0),$$

where $U_t^{(k)}$ and U_t are given by Eqs. (1) and (4), respectively.

Note that $T_t^{(k)}$ is intentionally skew with respect to U_t and $U_t^{(k)}$.

Lemma 2: *For each $k > 0$, the families of bounded linear maps $T_t^{(k)}$ $(t \geq 0)$ and T_t $(t \geq 0)$ given by Definition 2 are norm continuous one-parameter contraction semigroups with generators*

$$\begin{aligned} \mathscr{L}^{(k)}(X) &= K^\dagger X + X K^{(k)} + L_i^\dagger X L_i^{(k)}, \qquad X \in \mathcal{B}(\mathcal{H}), \\ \mathscr{L}(X) &= K^\dagger X + X K + L_i^\dagger X L_i, \qquad X \in \mathcal{B}(\mathcal{H}_0), \end{aligned} \tag{7}$$

respectively. That is $T_t^{(k)} = \exp\left(t\mathscr{L}^{(k)}\right)$ and $T_t = \exp(t\mathscr{L})$ for all $t \geq 0$.

Proof. We only prove the lemma for $T_t^{(k)}$. The proof for T_t can be obtained in an analogous way. Since the conditional expectation $\mathrm{id} \otimes \phi$ is norm contractive and U_t and $U_t^{(k)}$ are unitary, we have

$$\left\| T_t^{(k)}(X) \right\| \leq \left\| U_t^\dagger X U_t^{(k)} \right\| \leq \left\| U_t^\dagger \right\| \left\| U_t^{(k)} \right\| \|X\| = \|X\|,$$

for all $X \in \mathcal{B}(\mathcal{H})$. This proves that $T_t^{(k)}$ is a contraction for all $t \geq 0$. An application of the quantum Itô rule [15], together with the fact that vacuum expectations of stochastic integrals vanish, shows that

$$dT_t^{(k)}(X) = \mathrm{id} \otimes \phi \left(d\left(U_t^\dagger X U_t^{(k)} \right) \right) =$$
$$\mathrm{id} \otimes \phi \left(U_t^\dagger \left(K^\dagger X + X K^{(k)} + L_i^\dagger X L_i^{(k)} \right) U_t^{(k)} \right) dt = T_t^{(k)} \left(\mathscr{L}^{(k)}(X) \right) dt,$$

for all $X \in \mathcal{B}(\mathcal{H})$. That is, $T_t^{(k)} = \exp\left(t\mathscr{L}^{(k)}\right)$ is a one-parameter semigroup with generator $\mathscr{L}^{(k)}$. Furthermore, $\mathscr{L}^{(k)}$ is bounded

$$\left\| \mathscr{L}^{(k)}(X) \right\| \leq \left(\left\| K^\dagger \right\| + \left\| K^{(k)} \right\| + \left\| L_i^\dagger \right\| \left\| L_i^{(k)} \right\| \right) \|X\|,$$

which proves that $T_t^{(k)}$ is norm continuous. □

The proof of Theorem 2.1 relies heavily on the Trotter-Kato theorem [21, 16] in combination with an argument due to Kurtz [17]. We have taken the formulation of the Trotter-Kato theorem from [3, Thm 3.17, page 80], see also [10, Chapter 1, Section 6]. The formulation is more general than needed for the proof of Theorem 2.1.

Theorem 4.1: Trotter-Kato Theorem *Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}_0$ be a closed subspace of $\mathcal{B}$. For each $k \geq 0$, let $T_t^{(k)}$ be a strongly continuous one-parameter contraction semigroup on $\mathcal{B}$ with generator $\mathscr{L}^{(k)}$. Moreover, let T_t be a strongly continuous one-parameter contraction semigroup on $\mathcal{B}_0$ with generator $\mathscr{L}$. Let $\mathcal{D}$ be a core for $\mathscr{L}$. The following conditions are equivalent:*

1. *For all $X \in \mathcal{D}$ there exist $X^{(k)} \in \mathrm{Dom}\left(\mathscr{L}^{(k)}\right)$ such that*
$$\lim_{k\to\infty} X^{(k)} = X, \qquad \lim_{k\to\infty} \mathscr{L}^{(k)}\left(X^{(k)}\right) = \mathscr{L}(X).$$
2. *For all $0 \leq s < \infty$ and all $X \in \mathcal{B}_0$*
$$\lim_{k\to\infty} \left\{ \sup_{0\leq t\leq s} \left\| T_t^{(k)}(X) - T_t(X) \right\| \right\} = 0.$$

Proposition 1: *Let $T_t^{(k)}$ and T_t be the one-parameter semigroups on $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H}_0)$ defined in Definition 2, respectively. We have*
$$\lim_{k\to\infty} \left\{ \sup_{0\leq t\leq s} \left\| T_t^{(k)}(X) - T_t(X) \right\| \right\} = 0,$$
for all $X \in \mathcal{B}(\mathcal{H}_0)$ and $0 \leq s < \infty$.

Proof. The proof follows the line of the proof of [17, Theorem 2.2]. Lemma 2 shows that $T_t^{(k)} = \exp\left(t\mathscr{L}^{(k)}\right)$ and $T_t = \exp(t\mathscr{L})$ are norm continuous, and therefore also strongly continuous semigroups with generators given by Eq. (7). This means we satisfy the assumptions of the Trotter-Kato Theorem (Thm. 4.1) with $\mathcal{D} = \mathcal{B}(\mathcal{H}_0)$ and $\mathrm{Dom}\left(\mathscr{L}^{(k)}\right) = \mathcal{B}(\mathcal{H})$.

We can write $\mathscr{L}^{(k)}(X) = \mathscr{L}_0(X) + k\mathscr{L}_1(X) + k^2\mathscr{L}_2(X)$, $X \in \mathcal{B}(\mathcal{H})$, where (recall Assumption 2)
$$\mathscr{L}_0(X) = K^\dagger X + XB + L_i^\dagger X G_i, \quad \mathscr{L}_1(X) = XA + L_i^\dagger X F_i, \quad \mathscr{L}_2(X) = XY.$$
Let X be an element in $\mathcal{B}(\mathcal{H}_0)$ and let X_1 and X_2 be elements in $\mathcal{B}(\mathcal{H})$. We define $X^{(k)} = X + \frac{1}{k}X_1 + \frac{1}{k^2}X_2$. Collecting terms with equal powers in k, we find
$$\begin{aligned}\mathscr{L}^{(k)}\left(X^{(k)}\right) = {} & \left(\mathscr{L}_0(X) + \mathscr{L}_1(X_1) + \mathscr{L}_2(X_2)\right) + \\ & k\left(\mathscr{L}_1(X) + \mathscr{L}_2(X_1)\right) + \\ & k^2\left(\mathscr{L}_2(X)\right) + \\ & \frac{1}{k}\left(\mathscr{L}_0(X_1) + \mathscr{L}_1(X_2)\right) + \frac{1}{k^2}\left(\mathscr{L}_0(X_2)\right).\end{aligned}$$
Note that $\mathscr{L}_2(X) = 0$ as $X \in \mathcal{B}(\mathcal{H}_0)$ and $P_0 Y = 0$. Using the existence of Y_1^{-1}, we set
$$\begin{aligned} X_1 &= -\mathscr{L}_1(X) Y_1^{-1} P_1, \\ X_2 &= -\left(\mathscr{L}_0(X) + \mathscr{L}_1(X_1)\right) Y_1^{-1} P_1. \end{aligned}$$

Using the properties of Y_1^{-1} in Assumption 3, we obtain

$$\begin{aligned}\lim_{k\to\infty} \mathscr{L}^{(k)}\left(X^{(k)}\right) &= \lim_{k\to\infty}\left(\mathscr{L}(X) + \frac{1}{k}\left(\mathscr{L}_0(X_1) + \mathscr{L}_1(X_2)\right) + \frac{1}{k^2}\mathscr{L}_0(X_2)\right)\\ &= \mathscr{L}(X).\end{aligned}$$

The proposition then follows from the Trotter-Kato Theorem. □

Note that for all $v \in \mathcal{H}_0$, we can write $U_t v \otimes \Phi = P_0 U_t v \otimes \Phi$. This leads to

$$\begin{aligned}\left\|(U_t^{(k)} - U_t)v \otimes \Phi\right\|^2 &= \left\|(U_t^{(k)} - P_0 U_t)v \otimes \Phi\right\|^2\\ &= \left\langle v, \left(2I - T_t^{(k)}(P_0) - T_t^{(k)}(P_0)^\dagger\right) v\right\rangle.\end{aligned}$$

Here we have used that $\mathrm{id}\otimes\phi$ is a positive map, i.e. it commutes with the adjoint. Using Proposition 1 and noting that $\mathscr{L}(P_0) = 0$ by Lemmas 2 and 1, we see that Theorem 2.1 holds for all vectors in $\mathcal{H}_0 \otimes \mathcal{F}$ of the form $\psi = v \otimes \Phi$. We now need to extend this to all $\psi \in \mathcal{H}_0 \otimes \mathcal{F}$.

Let f be an element in $L^2(\mathbb{R}^+; \mathbb{C}^n)$. Denote by f_t the function f truncated at time t, i.e. $f_t(s) = f(s)$ if $s \le t$ and $f_t(s) = 0$ otherwise. Define the *Weyl operator* $W(f_t)$ as the unique solution to the following QSDE

$$dW(f_t) = \left\{ f(t)_i dA_t^{i\dagger} - \overline{f(t)}_i dA_t^i - \frac{1}{2}\overline{f(t)}_i f(t)_i dt \right\} W(f_t), \qquad W(f_0) = I. \tag{8}$$

Note that $W(f_t)$ is a unitary operator from $\mathcal{F}$ to $\mathcal{F}$. Moreover, it is not hard to see that $\pi(f_t) = W(f_t)\Phi$, see e.g. [20]. Often we will identify a constant $\alpha \in \mathbb{C}^n$ with the constant function on $\mathbb{R}^+$ taking the value α (truncated at some large $T \ge 0$ so that it is an element of $L^2(\mathbb{R}^+; \mathbb{C}^n)$).

Definition 3: Let f be an element in $L^2(\mathbb{R}^+; \mathbb{C}^n)$. Suppose that Assumptions 1, 2, 3 and 4 hold and let $U_t^{(k)}$ and U_t be given by Eqs. (1) and (4), respectively. Define

$$\begin{aligned}&U_t^{(kf)} = W(f_t)^\dagger U_t^{(k)} W(f_t), \qquad U_t^{(f)} = W(f_t)^\dagger U_t W(f_t),\\ &T_t^{(kf)}(X) = \mathrm{id}\otimes\phi\left(U_t^{(f)\dagger} X U_t^{(kf)}\right), \qquad X \in \mathcal{B}(\mathcal{H}),\\ &T_t^{(kf)}(X) = \mathrm{id}\otimes\phi\left(U_t^{(f)\dagger} X U_t^{(f)}\right), \qquad X \in \mathcal{B}(\mathcal{H}_0).\end{aligned}$$

Definition 4: Let α be an element in $\mathbb{C}^n$ and let i be an element in $\{1, \ldots, n\}$. Let $K^{(k)}, K, L_i^{(k)}, L_i, S_{ij}^{(k)}$ and S_{ij} be the coefficients of Eqs. (1) and (4). Define operators $K^{(k\alpha)}, K^{(\alpha)}, L_i^{(k\alpha)}$ and $L_i^{(\alpha)}$ by

$$\begin{aligned}&K^{(\alpha)} = K + \bar{\alpha}_i(S_{ij} - P_0\delta_{ij})\alpha_j + \bar{\alpha}_i L_i - \alpha_j L_i^\dagger S_{ij}, \qquad L_i^{(\alpha)} = L_i + \alpha_j S_{ij},\\ &K^{(k\alpha)} = K^{(k)} + \bar{\alpha}_i(S_{ij}^{(k)} - \delta_{ij})\alpha_j + \bar{\alpha}_i L_i^{(k)} - \alpha_j L_i^{(k)\dagger} S_{ij}, \quad L_i^{(k\alpha)} = L_i^{(k)} + \alpha_j S_{ij}^{(k)}.\end{aligned}$$

Note that with the coefficients given by Definition 4, applying the quantum Itô rule to $U_t^{(k\alpha)}$ and $U_t^{(\alpha)}$, defined in Definition 3, gives

$$
\begin{aligned}
dU_t^{(\alpha)} &= \left\{ \Big(S_{ij} - \delta_{ij}P_0\Big) d\Lambda_t^{ij} + L_i^{(\alpha)} dA_t^{i\dagger} - L_i^{(\alpha)\dagger} S_{ij} dA_t^j \; + K^{(\alpha)} dt \right\} U_t^{(\alpha)}, \\
dU_t^{(k\alpha)} &= \left\{ \Big(S_{ij}^{(k)} - \delta_{ij}\Big) d\Lambda_t^{ij} + L_i^{(k\alpha)} dA_t^{i\dagger} - L_i^{(k\alpha)\dagger} S_{ij}^{(k)} dA_t^j \; + K^{(k\alpha)} dt \right\} U_t^{(k\alpha)},
\end{aligned}
\tag{9}
$$

with $U_0^{(\alpha)} = U_0^{(k\alpha)} = I$.

Definition 5: Suppose that Assumptions 1, 2, 3 and 4 hold. Let α be an element in $\mathbb{C}^n$ and let i be an element in $\{1,\ldots,n\}$. Define operators $A^{(\alpha)}, B^{(\alpha)}$ and $G_i^{(\alpha)}$ by

$$
\begin{aligned}
A^{(\alpha)} &= A + F_i\bar{\alpha}_i - \alpha_j F_i^\dagger W_{ij}, \\
B^{(\alpha)} &= B + \bar{\alpha}_i(W_{ij} - \delta_{ij})\alpha_j + G_i\bar{\alpha}_i - \alpha_j G_i^\dagger W_{ij}, \\
G_i^{(\alpha)} &= G_i + \alpha_j W_{ij}.
\end{aligned}
$$

Lemma 3: *Suppose Assumptions 1, 2, 3 and 4 hold. Let $A, B, Y, F_i, G_i, W_{ij}, K, L_i$ and S_{ij} for $i,j \in \{1,\ldots,n\}$ be the various operators occuring in Assumption 1, 2, 3 and 4. Let $K^{(\alpha)}$ and $L_i^{(\alpha)}$ for $i \in \{1,\ldots,n\}$ be given by Definition 4 and let $A^{(\alpha)}, B^{(\alpha)}$ and $G_i^{(\alpha)}$ for $i \in \{1,\ldots,n\}$ be given by Definition 5. Then*

$$
L_i^{(\alpha)} = (G_i^{(\alpha)} - F_i Y_1^{-1} A^{(\alpha)}) P_0, \tag{10a}
$$

$$
K^{(\alpha)} = P_0 \left(B^{(\alpha)} - A^{(\alpha)} Y_1^{-1} A^{(\alpha)} \right) P_0, \tag{10b}
$$

i.e. Definition 1 holds with $A = A^{(\alpha)}$, $B = B^{(\alpha)}$, $G_i = G_i^{(\alpha)}$ $L_i = L_i^{(\alpha)}$ and $K = K^{(\alpha)}$. Moreover, Assumptions 1, 2, 3 and 4 hold for the altered coefficients with P_0 and Y_1^{-1} unchanged.

Proof. To show that Definition 1 holds for the altered coefficients, substitute $G_i^{(\alpha)}$ and $A^{(\alpha)}$ from Definition 5, and $L_i^{(\alpha)}$ from Definition 4 into Eq. (10a). This gives

$$
L_i + \alpha_j S_{ij} = \left(L_i + \alpha_j W_{ij} + \alpha_j F_i Y_1^{-1} F_l^\dagger W_{lj}, \right) P_0,
$$

which holds if we substitute $S_{ij} = \left(W_{ij} + F_i Y_1^{-1} F_l^\dagger W_{lj} \right) P_0$ from Definition 1. Furthermore, substituting $A^{(\alpha)}$ and $B^{(\alpha)}$ from Definition 5, and $K^{(\alpha)}$ from Definition 4 into Eq. (10b) gives

$$
\begin{aligned}
&\bar{\alpha}_i S_{ij}\alpha_j + \bar{\alpha}_i L_i - \alpha_j L_i^\dagger S_{ij} = P_0 \bar{\alpha}_i W_{ij}\alpha_j P_0 + P_0 G_i P_0 \bar{\alpha}_i - \alpha_j P_0 G_i^\dagger W_{ij} P_0 \\
&- P_0(F_i\bar{\alpha}_i + A) Y_1^{-1} (A - \alpha_j F_l^\dagger W_{lj}) P_0 + P_0 A Y_1^{-1} A P_0.
\end{aligned}
$$

This holds if we can show that

$$S_{ij} = P_0 \left(W_{ij} + F_i Y_1^{-1} F_l^\dagger W_{lj} \right) P_0 \tag{11a}$$

$$L_i = P_0 \left(G_i - F_i Y_1^{-1} A \right) P_0 \tag{11b}$$

$$L_i^\dagger S_{ij} = P_0 \left(G_i^\dagger W_{ij} - A Y_1^{-1} F_i^\dagger W_{ij} \right) P_0. \tag{11c}$$

Equations (11a) and (11b) are satisfied by Assumption 4 as $P_1 L_i = P_1 S_{ij} = 0$. Note that Eq. (11c) holds if we can show

$$L_i^\dagger \left(\delta_{il} + F_i Y_1^{-1} F_l^\dagger \right) W_{lj} P_0 = P_0 G_l^\dagger W_{lj} P_0 - P_0 A Y_1^{-1} F_l^\dagger W_{lj} P_0.$$

Substituting L_i from Definition 1, this becomes

$$\begin{aligned} & - P_0 A^\dagger Y_1^{-1\dagger} F_l^\dagger W_{lj} P_0 + P_0 G_i^\dagger F_i Y_1^{-1} F_l^\dagger W_{lj} P_0 \\ & \qquad - P_0 A^\dagger Y_1^{-1\dagger} F_i^\dagger F_i Y_1^{-1} F_l^\dagger W_{lj} P_0 + P_0 A Y_1^{-1} F_l^\dagger W_{lj} P_0 = 0. \end{aligned}$$

Now recall that $P_0(A + A^\dagger)P_1 = -P_0 G_i^\dagger F_i P_1$, and $Y + Y^\dagger = -F_i^\dagger F_i$ (see Eq. (3)) by Assumptions 1, 2 and 3. Moreover, $Y Y_1^{-1} P_1 F_l^\dagger W_{lj} P_0 = P_1 F_l^\dagger W_{lj} P_0$ by Assumption 3 which shows that Eq. (11c) is satisfied.

We now show that Assumptions 1, 2, 3 and 4 hold for the altered coefficients, with P_0 and Y_1^{-1} unchanged. Assumption 1 holds for the altered coefficients since, by Definition 3, we have $U_t^{(k\alpha)} = W(f_t)^\dagger U_t^{(k)} W(f_t)$ which is clearly unitary. By Assumption 2 for the original coefficients and Definition 4 and 5, we see that Assumption 2 holds for the altered coefficients. Assumption 3 on the altered coefficients is seen to hold by direct substitution of the coefficients in Definition 4 and 5, followed by application of Assumption 3 for the original system. Assumption 4 holds if $P_1 L_i^{(\alpha)} = P_1 L_i + \alpha_i P_1 S_{ij} = 0$, which follows from Assumption 4 on the original system. □

Lemma 3 shows that Proposition 1 holds with $T_t^{(k\alpha)}$ and $T_t^{(\alpha)}$ replacing $T_t^{(k)}$ and T_t, respectively.

Corollary 1: Suppose that Assumption 1, 2, 3 and 4 hold. Let α be an element of $\mathbb{C}^n$. We have

$$\lim_{k\to\infty} \left\{ \sup_{0\le t\le s} \left\| T_t^{(k\alpha)}(X) - T_t^{(\alpha)}(X) \right\| \right\} = 0,$$

for all $X \in \mathcal{B}(\mathcal{H}_0)$ and $0 \le s < \infty$.

Proof of Theorem 2.1. Let $t \ge 0$. Let f be a step function in $L^2([0,t];\mathbb{C}^n)$, i.e. there exists an $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_m = t$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}^n$ such that

$$s \in [t_{i-1}, t_i) \Longrightarrow f(s) = \alpha_i, \qquad \forall i \in \{1, \ldots, m\}.$$

The cocycle property of solutions to QSDE's and the exponential property of the symmetric Fock space lead to

$$T_t^{(kf)}(X) = T_{t_1}^{(k\alpha_m)} \dots T_{t-t_{m-1}}^{(k\alpha_1)}(X), \qquad X \in \mathcal{B}(\mathcal{H}),$$
$$T_t^{(f)}(X) = T_{t_1}^{(\alpha_m)} \dots T_{t-t_{m-1}}^{(\alpha_1)}(X), \qquad X \in \mathcal{B}(\mathcal{H}_0).$$

It is easy to see that Corollary 1 also holds for the difference of a finite product of maps $T_{t_i-t_{i-1}}^{(k\alpha_i)}$ and a finite product of maps $T_{t_i-t_{i-1}}^{(\alpha_i)}$. This leads to

$$\lim_{k\to\infty} \left\| T_t^{(kf)}(X) - T_t^{(f)}(X) \right\| =$$
$$\lim_{k\to\infty} \left\| T_{t_1}^{(k\alpha_m)} \dots T_{t-t_{m-1}}^{(k\alpha_1)}(X) - T_{t_1}^{(\alpha_m)} \dots T_{t-t_{m-1}}^{(\alpha_1)}(X) \right\| = 0, \qquad X \in \mathcal{B}(\mathcal{H}_0).$$

This immediately yields for all step functions $f \in L^2([0,t];\mathbb{C}^n)$ and $v \in \mathcal{H}_0$

$$\lim_{k\to\infty} U_t^{(k)} v \otimes \pi(f) = U_t v \otimes \pi(f). \tag{12}$$

Note that the step functions are dense in $L^2([0,t];\mathbb{C}^n)$. This means that Eq. (12) holds for all $f \in L^2([0,t];\mathbb{C}^n)$. Now note that for all $f \in L^2(\mathbb{R}^+;\mathbb{C}^n)$ and $t \le s \le \infty$, we have (e.g. [20])

$$W(f_s)^\dagger U_t^{(k)} W(f_s) = U_t^{(kf_t)}, \qquad W(f_s)^\dagger U_t W(f_s) = U_t^{(f_t)}.$$

This means that the result in Eq. (12) is true for all $f \in L^2(\mathbb{R}^+;\mathbb{C}^n)$. We now have

$$\lim_{k\to\infty} U_t^{(k)}\psi = U_t\psi,$$

for all ψ in $\mathscr{D} = \mathrm{span}\{v \otimes \pi(f);\ v \in \mathcal{H}_0, f \in L^2(\mathbb{R}^+;\mathbb{C}^n)\}$. Theorem 2.1 then follows from the fact that $\mathscr{D}$ is dense in $\mathcal{H}_0 \otimes \mathcal{F}$ (e.g. [20]).

5 Discussion

In this article we have studied adiabatic elimination in the context of the quantum stochastic models introduced by Hudson and Parthasarathy. We have shown strong convergence of a quantum stochastic differential equation to its adiabatically eliminated counterpart, under four assumptions. Physically, the first Assumption 1 enforces the unitarity of the initial QSDE model. Assumption 2 ensures an appropriate scaling in the coupling parameter k such that we can distinguish excited and ground states in our system. Assumptions 3 and 4 ensure the existence of a limit dynamics independent of k. Note that Assumption 4 specifically forbids any quantum jumps which terminate in an excited state, the presence of which would preclude the construction of a valid limit dynamics.

Although a Dyson series expansion for $U_t^{(k)}$ (e.g. in terms of Maassen kernels [18]) would provide a lot of intuition for the results we have obtained (see [14] and [3, Chapter 5, Section 4]), we have chosen a proof along the lines of semigroups and their generators.

An infinitesimal treatment has the advantage that it can exploit the existence of results such as the quantum Itô rule [15], the Trotter-Kato Theorem [21, 16] and the technique due to Kurtz [17].

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